Analytical solutions for the Kelvin-Helmholtz instability problem for a semi-infinite heterogeneous inviscid incompressible fluid with exponentially decreasing with the height density and a continuously variable as a linear function of height streaming velocity. Whittaker function zeros and other Whittaker function operator solutions.

Consider a semi-infinite layer of a heterogeneous inviscid incompressible fluid with exponentially decreasing density $\rho=\rho_0 e^{-\beta z}$ (z >=0 is the height in the layer), so that in the unperturbed state the fluid has a horizontal streaming velocity $U=U_0 z/d$ continuously variable as a linear function of the height (z>=0) where ρ_0,β,d,U_0 are constants, the first three being positive.

We consider small perturbations of the initial state given by actual density, velocity components and pressure at any point (x,y,z) in $\mathbb{R}^2\times\mathbb{R}_+$ and any moment of time t in $\mathbb{R}_+$, respective $(\rho+\delta\rho)(x,y,z,t)$, $(U+u,v,w)(x,y,z,t)$ and $(p+\delta p)(x,y,z,t)$ where $\delta\rho,u,v,w,\delta p$ are the small perturbations.

Then at first order (according [1]) the equations governing the perturbation are :

$$\rho\frac{\partial u}{\partial t}+\rho U\frac{\partial u}{\partial x}+\rho w\frac{dU}{dz}=\frac{-\partial}{\partial x}\delta p \qquad (1)$$

$$\rho\frac{\partial v}{\partial t}+\rho U\frac{\partial v}{\partial x}=\frac{-\partial}{\partial y}\delta p \qquad (2)$$

$$\rho\frac{\partial w}{\partial t}+\rho U\frac{\partial w}{\partial x}=\frac{-\partial}{\partial z}\delta p-g\delta\rho \qquad (3)$$

(where g is the gravitational acceleration), derived from the motions equations,

$$\frac{\partial}{\partial t}\delta\rho+U\frac{\partial}{\partial x}\delta\rho=-w\frac{\partial\rho}{\partial z} \qquad (4)$$

, derived from the continuity equation and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (5)$$

, the incompressibility condition.

Analysing the disturbance into normal modes we seek solutions of eq. (1)-(5) whose dependence on x,y,t is given by

$$\exp(i(k_x x+k_y y+nt)) \qquad (6)$$

where k_x , $k_y\in\mathbb{R}$ are the wave numbers and $n\in\mathbb{C}$ denotes the angular frequency.

It is obvious that if $n\in\mathbb{R}$ or $\Im n\geq 0$ the corresponding modes are stable, remaining small as t increases to infinity.

For such solutions, equations (1)-(5) become

$$i\rho(n+k_x U)u+\rho(D\,U)w=-ik_x\delta p \qquad (7)$$

$$i\rho(n+k_x U)v=-ik_y\delta p \qquad (8)$$

$$i\rho(n+k_x U)w=-D\,\delta p-g\,\delta\rho \qquad (9)$$

$$i(n+k_x U)\delta\rho=-w\,D\,\rho \qquad (10)$$

$$i(k_x u+k_y v)=-D\,w \qquad (11)$$

where D denotes d/dz.

Multiplying eq. (7) and (8) by $-ik_x$ and $-ik_y$ respectively, adding and using eq. (11) we obtain

$$i\rho(n+k_x U)D\,w-i\rho k_x(D\,U)w=-k^2\delta\rho \qquad (12)$$

where $k^2=k_x^2+k_y^2$

Combining eq. (9) and (1 0) we have

$$i\rho(n+k_x U)w=-D\,\delta p-ig(D\,\rho)\frac{w}{n+k_x U} \qquad (13)$$

Eliminating δp between (12) and (13) we obtain

$$D\,\{\rho(n+k_x U)D\,w-\rho k_x(D\,U)w\}-k^2\rho(n+k_x U)w=$$

$$=gk^2(D\,\rho)\frac{w}{n+k_x U} \qquad (14)$$

The solution must satisfy the boundary conditions

$w=0$ at $z=0$ and $z\to\infty$

Equation (14) can be written as

$$(n+k_x U)(D^2-k^2)w-k_x(D^2 U)w-gk^2\frac{D\rho}{\rho}\frac{w}{n+k_x U}+$$

$$+\frac{D\rho}{\rho}[(n+k_x U)Dw-k_x(DU)w]=0 \quad (15)$$

Without loss of generality we can assume that $k_x=k(U_0$ appears in (15) only as $k_x U_0)$ and so, measuring z in the unit d and taking

$n=U_0\nu/d$, $k=\kappa/d$, $\beta=\lambda/d$ and $J=g\beta d^2/U_0^2$, J being the non-dimensional Richardson number, equation (15) can be rewritten as

$$(\nu+\kappa z)^2\{D^2 w-\lambda D w+[-\kappa^2+\frac{\lambda\kappa}{\nu+\kappa z}+\frac{\kappa^2 J}{(\nu+\kappa z)^2}]w\}=0 \qquad (16)$$

with the boundary conditions $w=0$ for $z=0$ and $z\to\infty$

By change of variables

$w=e^{\lambda z/2}W$ and $\zeta=(z+\nu/\kappa)\sqrt{4\kappa^2+\lambda^2}$ equation (16) becomes

$$\zeta^2(\frac{d^2W}{d\zeta^2}+(\frac{-1}{4}+\frac{j}{\zeta}+\frac{\frac{1}{4}-m^2}{\zeta^2})W)=0 \qquad (17)$$

or $\zeta^2\mathscr{L}W=0$

where $j=\frac{\lambda}{\sqrt{4\kappa^2+\lambda^2}}\in(0\ ,\ 1)$ and $m^2=\frac{1}{4}-J<\frac{1}{4}$

Note that $\mathscr{L}=\frac{d^2}{d\zeta^2}-\frac{1}{4}+\frac{j}{\zeta}+\frac{\frac{1}{4}-m^2}{\zeta^2}$ is the Whittaker function operator.

The solution for W requires therefore (17) to be satisfied with boundary conditions

$$W((\frac{\nu}{\kappa})\sqrt{4\kappa^2+\lambda^2})=0 \qquad (17*)$$

$$w=\exp(\frac{1}{2}\lambda(\frac{1}{\sqrt{4\kappa^2+\lambda^2}}\zeta-\frac{\nu}{\kappa}))W=0 \qquad (17**)$$

for $\frac{1}{\sqrt{4\kappa^2+\lambda^2}}\zeta-\frac{\nu}{\kappa}=z\to\infty$

According [2], solutions of equation $\mathscr{L}W=0$ are linear combinations of Whittaker functions

$W_{j\ ,\ m}(z)$ and $W_{-j\ ,\ m}(-z)$ where for $j\ ,\ m\in\mathbb{C}$ with $j+\frac{1}{2}-m\notin\{0\ ,\ -1\ ,\ -2\ ,\ \ldots\}$

$$W_{j\ ,\ m}=\frac{-1}{2\pi i}\Gamma(j+\frac{1}{2}-m)e^{\frac{-1}{2}z}z^j\int_{\infty}^{(0+)}(-t)^{-j-\frac{1}{2}+m}(1+\frac{t}{z})^{j-\frac{1}{2}+m}e^{-t}dt \quad (18)$$

where arg z has it’s principal value and the contour from ∞ en-cycling 0+ directly and back to ∞ is chosen so that $t=-z$ is outside of it and let be taken $|\arg(-t)|\le\pi$ and also the value of $\arg(1+\frac{t}{z})$ which tends to zero as t tends to zero by a path lying inside the contour.

If $j+\frac{1}{2}-m\in\{0\ ,\ -1\ ,\ -2\ ,\ \ldots\}$ because $\Re(j-\frac{1}{2}-m)\le 0$ we can take

$$W_{j,m}=\frac{e^{\frac{-1}{2}z}z^{j}}{\Gamma(\frac{1}{2}-j+m)}\int_{0}^{\infty}t^{-j-\frac{1}{2}+m}(1+\frac{t}{z})^{j-\frac{1}{2}+m}e^{-t}dt \qquad (19)$$

and so $W_{j,m}(z)$ is defined for all $z\in\mathbb{C}\setminus\mathbb{R}_{-}$

For

$$M_{l,m}(z)=z^{\frac{1}{2}+m}e^{\frac{-1}{2}z}\left\{1+\frac{\sum_{p=1}^{\infty}\left(\frac{1}{2}+m-l\right)\dots\left(\frac{1}{2}+m-l+p-1\right)}{p!(2m+1)\dots(2m+p)}z^{p}\right\} \qquad (20)$$

according [2] , it can be shown that if $m\in\mathbb{C}$ and $2m\notin\mathbb{Z}$ and $|\arg z|<3\frac{\pi}{2}$ then

$$W_{l,m}(z)=\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-l)}M_{l,m}(z)+\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-l)}M_{l,-m}(z) \qquad (21)$$

and therefore $W_{l,m}=W_{l,-m} \qquad (22)$

if $m=si$ with $s\in\mathbb{R}^{*}$ and $l\in\mathbb{R}$ we have

$$\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-l)}=\left(\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-l)}\right)^{*}$$

and so if we take $\arg z^{*}=-\arg z$,for $2m\notin\mathbb{Z}$, $l\in\mathbb{R}$ we have $(W_{l,m}(z))^{*}=W_{l,m}(z^{*}) \qquad (23)$

It follows that for real z , if $m^{2}\in\mathbb{R}^{*}$, $2m\notin\mathbb{Z}$ and $l\in\mathbb{R}$ also $W_{l,m}(z)$ is real

$M_{j,m}$ also satisfies equation $\mathscr{L}W=0$ and we have, according [2] ,the Kummer second formula :

$$M_{0,m}(z)=z^{\frac{1}{2}+m}(1+\frac{\sum_{p=1}^{\infty}z^{2p}}{2^{4p}p!(m+1)\dots(m+p)}) \text{ for } 2m\in\mathbb{C}\setminus\mathbb{Z} \qquad (24)$$

$M_{j,m}$ and $W_{j,m}$ form a fundamental system of solutions for eq. $\mathscr{L}\,W=0$

Asymptotic expansions for $W_{l,m}(z)$ and $W'_{l,m}(z)$ when $|\arg z|<\pi-\alpha$,with $\alpha>0$

From relation (18) , because obviously we can derivate under the integral sign, it is easy to obtain a relation for the derivative $W'_{j,m}(z)$ and how is shown in [2] we can prove (by induction method) that

$$(1+\frac{t}{z})^{\lambda}=1+\frac{\lambda}{1!}\frac{t}{z}+\dots+\frac{\lambda(\lambda-1)\dots(\lambda-n+1)}{n!}\frac{t^{n}}{z^{n}}+R_{n}(t\ ,\ z)$$

where

$$R_{n}(t\ ,\ z)=\frac{\lambda(\lambda-1)\dots(\lambda-n)}{n!}(1+\frac{t}{z})^{\lambda}\int_{0}^{t/z}u^{n}(1+u)^{-\lambda-1}du$$

If $|\arg z| < \pi - \alpha$, $\alpha > 0$ and $|z| > 1$ it is easy to prove that

$$1 \le \left|1 + \frac{t}{z}\right| \le 1 + |t|$$

$$\left|1 + \frac{t}{z}\right| \ge \sin(\alpha) \text{ and therefore}$$

$$R_n(t, z) \le \left|\frac{\lambda(\lambda-1)\ldots(\lambda-n)}{n!}\right| (1+|t|)^{|\lambda|} (\operatorname{cosec})^{|\lambda|}(\alpha) \int_0^{|t/z|} u^n (1+u)^{|\lambda|} du \le$$

$$\le \left|\frac{\lambda(\lambda-1)\ldots(\lambda-n)}{n!}\right| (1+|t|)^{2|\lambda|} (\operatorname{cosec} \alpha)^{|\lambda|} \left|\frac{t}{z}\right|^{n+1} \frac{1}{n+1} = O(z^{-n-1})|t|^{n+1}$$

Integrating term by term in the expressions for

$W_{l,m}$ and $W'_{l,m}$ with the expansion of $\left(1 + \frac{t}{z}\right)^{\lambda}$ because

$$\int_{\infty}^{(0+)} (-t)^{-l-\frac{1}{2}+m} (t)^{n+1} O\left(z^{-n-1}\right) e^{-t} dt = -2i \sin\left(\pi\left(-l + \frac{1}{2} + m + n + 1\right)\right) (-1)^{n+1} \int_0^{\infty} t^{-l-\frac{1}{2}+m+n+1} e^{-t} O\left(z^{-n-1}\right) dt =$$

$$= O\left(z^{-n-1}\right) \text{ and}$$

$$\int_0^{\infty} t^{-l-\frac{1}{2}+m+n+1} e^{-t} dt = \Gamma\left(-l + \frac{1}{2} + m + n + 1\right) \text{ when } n \text{ is sufficiently large for that}$$

$$\Re\left(-l + \frac{1}{2} + m + n + 1\right) > 0$$

follows the asymptotic behaviour for

$W_{l,m}(z)$ and $W'_{l,m}(z)$ when $|z| \to \infty$, $|\arg z| \le \pi - \alpha$ with $\alpha > 0$:a

$$W_{l,m}(z) = e^{\frac{-1}{2}z} z^l \left(1 + O(z^{-1})\right) \text{ and}$$

$$W'_{l,m}(z) = \frac{-1}{2} e^{\frac{-1}{2}z} z^l \left(1 + O(z^{-1})\right)$$

where we have considered that

$$\frac{-1}{2\pi i} \int_{\infty}^{(0+)} (-t)^{-l-\frac{1}{2}+m} e^{-t} dt = \frac{1}{\Gamma(l + \frac{1}{2} - m)}$$

Some solutions are given by functions which satisfy the Whittaker equation $\mathscr{L} W = 0$, the boundary condition at infinity (17**) and the dispersion relation between the wave number and the angular velocity is given by (17*). Considering the asymptotic behaviour , such solutions are given by multiples of the Whittaker function $w_{j,m}$ and the dispersion relation comes from the zeros of the Whittaker function.

We will show that in the considered situation, namely j and $\frac{1}{4} - m^2$ being real the Whittaker function has only real zeros and therefore the corresponding modes of perturbation, if exist, are stable modes.

Let $W = W_{j,m}$. Suppose that $W(z_0) = 0$, $z_0 = x_0 + i y_0$, $y_0 \ne 0$

Because the zeros of W are isolated points in $\mathbb{C} \setminus \mathbb{R}_-$ (W is holomorphic on $\mathbb{C} \setminus \mathbb{R}_-$) we can chose $x_0 = \max\left\{x \in \mathbb{R} \mid W(x, y_0) = 0\right\}$

Let

$$W = p + iq$$

$$Q(z) = \frac{-1}{4} + \frac{j}{z} + \frac{\frac{1}{4} - m^2}{z^2}$$

$p\ ,\ q$ satisfy Cauchy-Riemann

$$\frac{\partial p}{\partial x}=\frac{\partial q}{\partial y}\ ,\ \frac{\partial p}{\partial y}=\frac{-\partial q}{\partial x}$$

and $W''=-QW$, the Whittaker equation which leads to:

$$\frac{\partial^2}{\partial x\partial y}(p^2+q^2)(x\ ,\ y)=\Im Q\,(p^2+q^2)(x\ ,\ y)$$

Let

$$\bar{h}=p^2+q^2\ ,\ u=x+y\ ,\ v=x-y$$

$h(u\ ,\ v)=\bar{h}\left(\frac{u+v}{2}\ ,\ \frac{u-v}{2}\right)$ and we have

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial u}+\frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial u}-\frac{\partial}{\partial v}$$

$$\frac{\partial^2}{\partial x\partial y}=\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}$$

We have

$$\frac{\partial^2 h}{\partial u^2}-\frac{\partial^2 h}{\partial v^2}=\Im Q\,h$$

Consider a cone with top at arbitrary

$(u_0\ ,\ v_0)\in\mathbb{R}^2\setminus D$ and height ϵ where

$$D=\left\{(u\ ,\ v)\in\mathbb{R}^2\middle|\frac{u+v}{2}+i\,\frac{u-v}{2}\in\mathbb{C}\setminus\mathbb{R}_-\right\}$$

$$B=B(u_0\ ,\ v_0\ ,\ \epsilon)=\left\{(u\ ,\ v)\in D\,\middle|\,u>u_0-\epsilon\ ,\ u+v<u_0+v_0\ ,\ v-u>v_0-u_0\right\}$$

Therefore

$$\iint_B \frac{\partial h}{\partial u}\Im Q\,h\,dvdu=\iint_B \frac{1}{2}\frac{\partial}{\partial u}\left[\left(\frac{\partial h}{\partial u}\right)^2\right]+\frac{1}{2}\frac{\partial}{\partial u}\left[\left(\frac{\partial h}{\partial v}\right)^2\right]-\frac{\partial}{\partial v}\left(\frac{\partial h}{\partial u}\frac{\partial h}{\partial v}\right)dvdu=$$

$$=\oint_{\partial B}\frac{1}{2}\left(\frac{\partial h}{\partial u}\right)^2\nu_u+\frac{1}{2}\left(\frac{\partial h}{\partial v}\right)^2\nu_u-\frac{\partial h}{\partial u}\frac{\partial h}{\partial v}\,\nu_v\,d\,\sigma=$$

$$=\frac{1}{2}\int_{v_0-\epsilon}^{v_0+\epsilon}\left(\left(\frac{\partial h}{\partial u}\right)^2+\left(\frac{\partial h}{\partial v}\right)^2\right)(u_0-\epsilon\ ,\ v)\,dv+\frac{1}{2\sqrt{2}}\int_{v_0}^{v_0+\epsilon}\left(\frac{\partial h}{\partial u}-\frac{\partial h}{\partial v}\right)^2(v_0+u_0-v\ ,\ v)\,dv+$$

$$+\frac{1}{2\sqrt{2}}\int_{v_0-\epsilon}^{v_0}\left(\frac{\partial h}{\partial u}+\frac{\partial h}{\partial v}\right)^2(v+u_0-v_0\ ,\ v)\,dv\geq 0$$

and we conclude , because B is an arbitrary chosen cone in the domain and h is positive, that

$$\Im Q\,\frac{\partial h}{\partial u}(u\ ,\ v)\geq 0 \text{ for any } (u\ ,\ v)\in D \qquad (25)$$

which is equivalent to

$$\Im Q\left(\frac{\partial\bar{h}}{\partial x}+\frac{\partial\bar{h}}{\partial y}\right)(x\ ,\ y)\geq 0 \text{ for any } x+i\,y\in\mathbb{C}\setminus\mathbb{R}_- \qquad (26)$$

It is easy to see that

$\Im Q(x\ ,\ y)=\Im Q(x\ ,-y)$ and that according (23)

$\bar{h}(x\ ,\ y)=\bar{h}(x\ ,-y)$ so that from (26) follows

$$\Im Q\left(\frac{\partial \bar{h}}{\partial y}-\frac{\partial \bar{h}}{\partial x}\right)(x\ ,\ y)\geq 0 \text{ for any } x+iy\in\mathbb{C}\setminus\mathbb{R}_{-} \quad (27)$$

(26) and (27) lead to

$$\Im Q\frac{\partial \bar{h}}{\partial y}(x\ ,\ y)\geq 0 \text{ for any } x+iy\in\mathbb{C}\setminus\mathbb{R}_{-} \quad (28)$$

From

$$\frac{\partial^2 p}{\partial x^2}=-\Re Q p+\Im Q q$$

$$\frac{\partial^2 q}{\partial x^2}=-\Im Q p-\Re Q q$$

which follows from the Whittaker equation, taking

$\bar{p}(s)=p(x_0+s\ ,\ y_0)$ and $\bar{q}(s)=q(x_0+s\ ,\ y_0)$

multiplying the first and second equation by q respectively by -p and adding them, we obtain

$$(\bar{p}'\bar{q}-\bar{q}'\bar{p})'(s)=(\bar{p}^2+\bar{q}^2)(s)\Im Q(x_0+s\ ,\ y_0) \quad (29)$$

Integrating (29) after s from 0 to infinity, considering the asymptotic behaviour of W and W', we have

$$\int_0^\infty (\bar{p}^2+\bar{q}^2)(s)\Im Q(x_0+s\ ,\ y_0)ds=0 \quad (30)$$

Because the equation in x variable

$\Im Q(x\ ,\ y_0)=0$ is polynomial of second degree, from (30) follows that we can consider

$$x_1=\max\{x\in\mathbb{R}\,|\,x>x_0\ ,\ \Im Q(x\ ,\ y_0)=0\} \quad (31)$$

so that $\Im Q(\ .\ ,\ y_0)$ changes sign at $x=x_1$

From (28) follows now that $\frac{\partial \bar{h}}{\partial y}(\ .\ ,\ y_0)$ changes sign at $x=x_1$ and so

$$\frac{\partial \bar{h}}{\partial y}(x_1\ ,\ y_0)=0 \quad (32)$$

Taking now

$\bar{p}(s)=p(x_1+s\ ,\ y_0)$ and $\bar{q}(s)=q(x_1+s\ ,\ y_0)$,

from (32) , considering the Cauchy-Riemann equations satisfied by p and q we obtain

$(\bar{p}'\bar{q}-\bar{q}'\bar{p})(0)=0$ and so , as above we have existing

$x_2=\max\{x\in\mathbb{R}\,|\,x>x_1\ ,\ \Im Q(x\ ,\ y_0)\}$, but this contradicts the choosing of x_1 according (31)

Therefore $y_0=0$ and the zeros of the Whittaker function must be real.

Note that the condition $y_0\neq 0$ plays a role in placing z_0 in $\mathbb{C}\setminus\mathbb{R}_{-}$ and making

$\Im Q(\ .\ ,\ y_0)$ a second degree polynomial.

For the following we mention

1. Sturm separation theorem

   Suppose that y_1 and y_2 are a fundamental pair of solutions (and hence are linearly independent) of the equation $y'' + qy = 0$ where q is a continuous function , on a interval I. Then :

   i) The zeros of non-trivial solutions of equation $y'' + qy = 0$ are isolated

   ii) If $x_1 < x_2$ are two consecutive zeros of y_1 then y_2 has exactly one zero in (x_1 , x_2)

2. Sturm comparison theorem

   Let y_1 and y_2 be non-trivial solutions of equations

   $y'' + q_1 y = 0$ and $y'' + q_2 y = 0$ where q_1 and q_2 are continuous on a interval I , such that $q_1 <= q_2$

   Then between two consecutive zeros x_1 and x_2 of y_1 , there exists at least one zero of y_2 unless

$q_1 = q_2$ on (x_1, x_2).

The zeros of W I.

Zeros of W - the case $m^2>0$

Because $W_{j,m}=W_{j,-m}$ we can take $m>0$ and because $m^2=\frac{1}{4}-J$, $J>0$

we have $m<\frac{1}{2}$

$W_{j,m}$ and $M_{j,m}$ are real functions , solutions on $\mathbb{R}_+$ of the equation $W''+QW=0$

Also we have

$$M_{j,m}(x)=e^{\frac{-1}{2}x}x^j\left(1+\left(\frac{1}{2}+m-j\right)g(x)\right) \text{ where for } j\in(0,1) , m>0$$

g is a power series with positive coefficients , with $g(0)=0$ and so g is a strictly increasing function on $\mathbb{R}_+$

Therefore

$M_{j,m}$ has at most one zero on $\mathbb{R}_+$;it has one zero if $\frac{1}{2}+m-j<0$ and none if $\frac{1}{2}+m-j\geq 0$

According to Sturm separation theorem W has at most one zero if $\frac{1}{2}+m-j\geq 0$ and at most two zeros if $\frac{1}{2}+m-j<0$

The zeros of W II.

Zeros of W - the case $m^2<0$, $m=is$, $s\in\mathbb{R}$

Again, because $W_{j,m}=W_{j,-m}$ we can take $s>0$

Consider the Sturm comparison theorem for

$$q_1=\frac{-1}{4}+\left(\frac{1}{4}-m^2\right)\frac{1}{x^2} \quad q_2=\frac{-1}{4}+\frac{j}{x}+\left(\frac{1}{4}-m^2\right)\frac{1}{x^2}$$

$y_1=\Re M_{0,m}(x)$, $y_2=W_{j,m}(x)$ on $x>0$

We observe that according Kummer’s second formula (see [2]) we have for x>0

$$2\Re M_{0,m}(x)=x^{\frac{1}{2}}\left(e^{is\ln(x)}+e^{-is\ln(x)}\right)+$$
$$+x^{\frac{1}{2}}\sum_{p=1}^{\infty}x^{2p}e^{is\ln(x)}\left(\frac{1}{2^{4p}p!(m+1)\ldots(m+p)}+e^{-2is\ln(x)}\frac{1}{2^{4p}p!((m+1)\ldots(m+p))^*}\right)$$

we take $x_n\downarrow 0$ with $s\ln(x_n)=-\left(n+\frac{1}{2}\right)\pi$ and it follows for $n\rightarrow\infty$ that we have

$$2\Re M_{0,m}\simeq(-1)^{n+1}x_n^{\frac{5}{2}}\frac{s}{8(1+s^2)}$$ and therefore

$\Re M_{0,m}(x_{2n})<0$ and $\Re M_{0,m}(x_{2n+1})>0$ for n sufficiently large

Thus, when $x_n\downarrow 0$ between x_{2n} and x_{2n+1} exists at least one zero of $\Re M_{0,m}$ and concluding with the comparison theorem, W has an at least countable infinite decreasing sequence of zeros.

From these results it follows that, when $J>\frac{1}{4}\ ,\ m^2<0$ there exists for each wave number k an infinite countable set of distinct real characteristic values for the angular velocity n corresponding to stable oscillatory modes, and when $J<\frac{1}{4}\ ,\ m^2>0$ for each wave number one or two real characteristic values corresponding to stable oscillatory modes may exist. Therefore the solution for the characteristic value problem does not lead to a complete set of proper values and proper functions derived from the equation $\mathscr{L}W=0\ \text{ if }\ J<\frac{1}{4}$.

To get a complete set of proper functions we must consider other solutions of equation $\zeta^2\mathscr{L}W=0$ which must also satisfy $w(z)=0\ \text{ for }\ z=0\ \text{ and }\ w(z)=0\ \text{ for }\ z\to\infty$.

Therefore we will seek for solutions W which satisfy $\zeta^2\mathscr{L}W=0$ in the sense of distributions.

When $|\arg\ \zeta|<\frac{3\pi}{2}\ \text{ and }\ 2m\notin\mathbb{Z}\ ,\ W_{\pm j\ ,\ m}(\zeta)$ can be defined by (21) where $M_{\pm j\ ,\ m}(\zeta)$ is defined by (20).

Let

$$\tilde{W}(\zeta)=\begin{cases} M_{j\ ,\ m}(\zeta)\ \text{ for }\ \zeta<0 \\ W_{j\ ,\ m}(\zeta)\ \text{ for }\ \zeta>0\end{cases}$$

For any arbitrary test function $\phi\in C_0^\infty(\mathbb{R})$ we have, with $\tilde{W}$ as distribution :

$$(\zeta^2\mathscr{L}\tilde{W}\ ,\ \phi)=\int_{\mathbb{R}}(\zeta^2\phi)''\tilde{W}d\zeta+\int_{\mathbb{R}}\zeta^2QW\phi d\zeta \qquad (33)$$

For $m^2<\frac{1}{4}$ it is easy to verify that

$$\lim_{\zeta\to 0}(\zeta^2\phi)'\tilde{W}=0$$
$$\lim_{\zeta\to 0}(\zeta^2\phi)\tilde{W}'=0$$

and that for $\zeta\neq 0$ we have $\tilde{W}''+Q\tilde{W}=0$

Hence integrating by parts in (33) it follows that $(\zeta^2\mathscr{L}\tilde{W}\ ,\ \phi)=0\ \text{ for any test function }\ \phi$.

For the same reason $\zeta^2\mathscr{L}(\tilde{W}+AW_{j\ ,m})=0$

as a distributions equality.

Thus for any non-zero $\zeta_0<0\ \text{ of }\ W_{j\ ,\ m}M_{j\ ,\ m}$

(note that the zeros of $W_{j\ ,\ m}M_{j\ ,\ m}$ are isolated points in $\mathbb{R}_-^*$

being zeros of a holomorphic function) we can take $A\in\mathbb{R}^*$ such that $(\tilde{W}+AW_{j\ ,m})(\zeta_0)=0$

Moreover, if m is real (i.e. $m^2>0$, and we can take m >0) we have that $\tilde{W}+AW_{j\ ,m}$ is not identical zero for real non-zero A, because from the definitions follows :

$$\lim_{z \uparrow 0} \left| z^{\frac{-1}{2}+m} A W_{j,m}(z) \right| = \left| A \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-j)} \right| \neq 0$$

and $\lim_{z \uparrow 0} z^{\frac{-1}{2}+m} \tilde{W}(z) = 0$

and we have the non-trivial solution $W = \tilde{W} + A W_{j,m}$

which for the wave number $k = \frac{\kappa}{d}$ and

arbitrary angular velocity $n = U_0 \frac{\nu}{d}$ with $\nu = \zeta_0 \frac{\kappa}{\sqrt{4\kappa^2 + \lambda^2}}$, $\zeta_0 < 0$ gives the stable oscillatory mode

$w = e^{\lambda \frac{z}{2}} W\left(\zeta_0 + z\sqrt{4\kappa^2 + \lambda^2}\right) e^{i(kx+nt)}$ which satisfies obviously the required boundary conditions at zero and infinity (because of the choice of A and the asymptotic behaviour of the Whittaker function).

We observe that ζ_0 passes all the strictly negative numbers without several isolated zeros how we noticed above and therefore the characteristic values to the corresponding modes form reunions of real continua and the conclusion is that for all Richardson numbers J an initial small perturbation becomes becomes a sum of oscillatory terms (derived from the solutions of $\mathscr{L} W = 0$) which necessarily exists if J>1/4 and a term corresponding to the real continua, which is necessarily non-trivial if J<1/4 and these perturbations remain small as time increases to infinity.
That was also the conclusion in [1]

References

[1] S. Chandrasekhar HYDRODYNAMIC AND HYDROMAGNETIC STABILITY,
[2] E.T. Whittaker, G.N. Watson A COURSE OF MODERN ANALYSIS