Early Earth simple model

Consider a sphere of incompressible homogeneous fluid, subject to a spherically symmetric radial gravitational field $g(r)x_i$, $r=\|x\|$, $x=(x_i)_{i=1,2,3,}$

At the initial state (with no motions) the fluid has a temperature radial distribution $T_0=T_0(x)$ and pressure $p_0=p_0(x)$ and constant density ρ_0

By considering the stability of this initial state we will have small perturbations of velocity $u=(u_i)_{i=1,2,3,}$, temperature θ , pressure δp and density $\delta\rho$, so that in the perturbed state we have $p=p_0+\delta p$, $T=T_0+\theta$, $\rho=\rho_0+\delta\rho$

We note that in the following we use the Einstein summation convention, so that if an index variable repeats itself in a term, then instead of that term we take the sum of all terms corresponding to the index variable, as the respective index passes all its values.

We have the Navier – Stokes equations of the fluid :

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\frac{\mu}{\rho}\nabla^2 u_i-g(r)x_i \qquad (1)$$

where μ is the coefficient of viscosity,

the caloric internal energy relation : $\dot{e}=c_v\dot{T}$ $\quad(2)$,

where c_v is the specific heat at constant volume and the dot on variables means that their total derivatives with respect to time variable t or the operator we denote $\frac{d}{dt}$ are taken,

the heat conduction equation : $q_i=-k\frac{\partial T}{\partial x_i}$ $\quad(3)$ where $q=(q_i)_{i=1,2,3}$ is the heat flux vector and k is the Fourier heat conduction coefficient,

the energy equation :

$$\rho\frac{de}{dt}=\boldsymbol{T}_{ij}\frac{\partial u_i}{\partial x_j}-\frac{\partial q_i}{\partial x_i}+\varepsilon \qquad (4)$$

, with $\frac{d}{dt}=\frac{\partial}{\partial t}+u_i\frac{\partial}{\partial x_i}$ the total derivative operator and ε a constant uniform distribution of heat sources which at initial state maintains a radial temperature gradient and with $\boldsymbol{T}$ the Cauchy tensor of surface tensions in the fluid.

For an incompressible fluid we have

$$\boldsymbol{T}_{ij}=-p\,\delta_{ij}+\frac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \qquad (5)$$

We suppose also $\rho=\rho_0(1-\alpha\theta)$ $\quad(6)$ where α is the coefficient of volume expansion and according to Boussinesq approximation we treat ρ as a constant $\rho=\rho_0$ in all terms of the equations of motion, except the ones in which the gravitational external force is present, when (6) is considered to be valid.

Further we consider that the perturbations and whose derivatives are small first order approximation and we will ignore higher order approximation terms.

Hence from (2) , (3) and (4) we have :

$$\frac{\partial\theta}{\partial t}=\kappa\nabla^2 T+\epsilon-u_i\frac{\partial T}{\partial x_i} \qquad (7)$$

, where $\kappa=\frac{k}{\rho_0 c_v}$ and $\epsilon=\frac{\varepsilon}{\rho_0 c_v}$

In the unperturbed initial state we have : $\kappa\nabla^2 T_0+\epsilon=0$ $\quad(8)$

and so, since T_0 is radial we have $T_0=\beta_0-\beta r^2$

with β_0 , β constants $\beta=\frac{\epsilon}{6\kappa}$

From (7) and (8) follows now at first order approximation

$$\frac{\partial\theta}{\partial t}=\kappa\nabla^2\theta+2\beta u_i x_i \qquad (9)$$

Since in the initial state $\rho_0 g(r)x_i+\frac{\partial p_0}{\partial x_i}=0 \quad (10)$, multiplying (1) with ρ and subtracting (10) , under the considered approximations we have

$$\frac{\partial u_i}{\partial t}=-\frac{1}{\rho_0}\frac{\partial\delta p}{\partial x_i}-\frac{\delta\rho}{\rho_0}g(r)x_i+\nu\nabla^2 u_i$$ or with (6) :

$$\frac{\partial u_i}{\partial t}=-\frac{1}{\rho}\frac{\partial\delta p}{\partial x_i}+\alpha\theta g(r)x_i+\nu\nabla^2 u_i \qquad (11)$$ where $\nu=\frac{\mu}{\rho_0}$ is the kinematic viscosity.

Also we have, from the continuity equation, for the incompressible fluid :

$$\frac{\partial u_i}{\partial x_i}=0 \qquad (12)$$

Let $\gamma(r)=\alpha g(r)$

Taking the curl of equation (11) we obtain

$$\frac{\partial\omega_i}{\partial t}=\gamma\epsilon_{ijk}\frac{\partial\theta}{\partial x_j}x_k+\nu\nabla^2\omega_i \qquad (13)$$, where $\omega=\nabla\times u$ is the vorticity field.

Taking the curl of this equation once again we have :

$$\frac{\partial}{\partial t}\nabla^2 u_i=-O_i\theta+\nu\nabla^4 u_i \qquad (14)$$ where O_i stands for the differential operator

$$O_i=-\epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{klm}\gamma x_l\frac{\partial}{\partial x_m}=\frac{\partial}{\partial x_j}\gamma\left(x_j\frac{\partial}{\partial x_i}-x_i\frac{\partial}{\partial x_j}\right)=$$

$$=\gamma\left(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial x_i}x_j\frac{\partial}{\partial x_j}-x_i\nabla^2\right)+\frac{1}{r}\frac{\partial\gamma}{\partial r}\left(r^2\frac{\partial}{\partial x_i}-x_i x_j\frac{\partial}{\partial x_j}\right)$$

Now we can directly verify that

$x_i\nabla^2 f_i=\nabla^2(x_i f_i)$ if $\frac{\partial f_i}{\partial x_i}=0$ (f is solenoidal) and since u , ω are both solenoidal, from (13) , (14) follows

$$\frac{\partial}{\partial t}(x_i\omega_i)=\nu\nabla^2(x_i\omega_i) \qquad (15)$$

$$\nabla^2\left(\nu\nabla^2-\frac{\partial}{\partial t}\right)(u_i x_i)=\gamma L^2\theta \qquad (16)$$ where $\gamma L^2=x_i O_i$

It is easy to verify that

$-L^2=(x\times\nabla)\cdot(x\times\nabla)$, $-L^2$ is the square of angular momentum operator and in spherical polar coordinates r , ϑ , ϕ we have

$$L^2=r^2\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\nabla^2\right)=-\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}-\frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\phi^2}$$

L^2 is the spherical harmonics operator and his eigenvalues are the spherical harmonics

$$Y_l^m(\vartheta,\phi)=P_l^m(\cos(\vartheta))e^{im\phi} \qquad (17)$$

$L^2 Y_l^m=l(l+1)Y_l^m \qquad (18)$ for $l\in\mathbb{N}$, $m\in\mathbb{Z}$, $|m|\le l$ where P_l^m are the associated Legendre polynomials and

$$\int_0^{\pi}\int_0^{2\pi}(Y_l^m(\vartheta,\phi))^2\sin(\vartheta)d\phi d\theta=N_l^m=\frac{4\pi}{2l+1}\frac{(l+m)!}{(l-m)!} \qquad (19)$$ is the normalization integral.

We can eliminate θ from the eq. (16) observing that $\nabla^2 L^2=L^2\nabla^2$ using eq. (9) and we obtain :

$$\left(\kappa\nabla^2-\frac{\partial}{\partial t}\right)\gamma^{-1}\nabla^2\left(\nu\nabla^2-\frac{\partial}{\partial t}\right)(u_i x_i)=-2\beta L^2(u_i x_i) \qquad (20)$$

We analyse the disturbance into normal modes, in terms of spherical harmonics.
Accordingly we will have :

$$x_i \omega_i = r\,\omega_r = Z(r) Y_l^m(\vartheta, \phi) \mathrm{e}^{nt}$$

$$x_i u_i = r\,u_r = W(r) Y_l^m(\vartheta, \phi) \mathrm{e}^{nt}$$

$$\theta = \Theta(r) Y_l^m(\vartheta, \phi) \mathrm{e}^{nt}$$

where n is a constant pulsation number which can be complex.

We define the operator $\mathscr{L}_l = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}$

Since $\nabla^2 Y_l^m(\vartheta, \phi) f(r) = Y_l^m(\vartheta, \phi) \mathscr{L}_l f(r)$ for any function f of r , the equations (9), (15), (16) lead to :

$$(\mathscr{L}_l - p\,\sigma)\Theta = \frac{-2\beta}{\kappa} R^2 W \qquad (21)$$

$$(\mathscr{L}_l - \sigma) Z = 0 \qquad (22)$$

$$\mathscr{L}_l(\mathscr{L}_l - \sigma) W = \frac{\gamma}{\nu} R^4 l(l+1)\Theta \qquad (23)$$

where we have measured r in units of sphere radius R and $\sigma = nR^2/\nu$, $p = \nu/\kappa$.

At the boundary of the sphere the perturbations of radial velocity and temperature must vanish and so we must require

$W=0$ and $\Theta=0$ for $r=1$

Also, when the surface of the sphere is free , and this is the case of our model, we must require that the tangential viscous stresses $\boldsymbol{T}_{r\vartheta}$ and $\boldsymbol{T}_{r\phi}$ vanish at $r=1$.

The expressions for these stress tensor components are according to (5) in spherical polar coordinates :

$$\boldsymbol{T}_{r\vartheta} = \rho\,\nu\left(\frac{1}{r}\frac{\partial u_r}{\partial \vartheta} - \frac{u_\vartheta}{r} + \frac{\partial u_\vartheta}{\partial r}\right)$$

$$\boldsymbol{T}_{r\phi} = \rho\,\nu\left(\frac{1}{r\sin(\vartheta)}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r}\right)$$

Since $u_r=0$ on $r=1$ it follows that on $r=1$ we have

$$r\frac{\partial}{\partial r}\left(\frac{u_\vartheta}{r}\right) = 0 \qquad (24)$$

$$r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right) = 0 \qquad (25)$$

Also we have the equation of continuity in spherical polar coordinates :

$$\frac{\partial u_r}{\partial r} + 2\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta} + \frac{u_\vartheta}{r}\cot(\vartheta) + \frac{1}{r\sin(\vartheta)}\frac{\partial u_\phi}{\partial \phi} = 0 \qquad (26)$$

Applying $r\frac{\partial}{\partial r}$ on (26), from (25) and (26) follows

$\frac{\partial^2}{\partial r^2}(r\,u_r) - \frac{2}{r}u_r$ at $r=1$ and since u_r vanishes at $r=1$ we have

$$\frac{d^2 W}{dr^2} = 0 \quad \text{at} \quad r=1 \qquad (27)$$

According to toroidal-poloidal decomposition , any solenoidal field U can be written as the sum of a toroidal field T and a polodal field S with existing scalar fields Ψ and Φ such that

$$U = T + S$$

$$T = \nabla \times \left(\frac{\Psi}{r} x\right) \text{ which is the same as } T = \left(\nabla \frac{\Psi}{r}\right) \times x$$

$$S = \nabla \times \left(\nabla \times \left(\frac{\Phi}{r} x\right)\right) \text{ which is the same as } S = \nabla \times \left(\left(\nabla \frac{\Phi}{r}\right) \times x\right)$$

In spherical polar coordinates we have :

$$T_r = 0 \ , \ T_\vartheta = \frac{1}{r\sin(\vartheta)} \frac{\partial \Psi}{\partial \phi} \ , \ T_\phi = -\frac{1}{r} \frac{\partial \Psi}{\partial \vartheta} \qquad (28)$$

$$S_r = \frac{1}{r^2} L^2 \Phi \ , \ S_\vartheta = \frac{1}{r} \frac{\partial^2 \Phi}{\partial r \partial \vartheta} \ , \ S_\phi = \frac{1}{r\sin(\vartheta)} \frac{\partial^2 \Phi}{\partial r \partial \phi} \qquad (29)$$

Since the velocity field is solenoidal, for the normal modes we have :

$$r u_r = \frac{l(l+1)}{r} \bar{S}(r) Y_l^m$$

$$r \omega_r = \frac{1}{r} l(l+1) \bar{T}(r) Y_l^m$$

where $\bar{T} Y_l^m$, $\bar{S} Y_l^m$ are the defining scalar fields of the toroidal respective poloidal fields from the decomposition of the velocity field.

Hence $\bar{S} = \frac{rW}{l(l+1)}$, $\bar{T} = \frac{rZ}{l(l+1)}$ $\qquad (30)$

<u>The gravitational field</u>

Since the fluid sphere is homogeneous, the gravitational mass force force field is given by

$$F(x) = -\int_B G\rho \frac{x-\xi}{\|x-\xi\|^3} dV(\xi)$$

where G is the gravitational constant, B is the ball of radius R and dV is the volume element.
The potential $\boldsymbol{V}$ is given by the volume potential

$\boldsymbol{V}(x) = \int_B \frac{G\rho}{\|x-\xi\|} dV(\xi)$ with $\nabla \boldsymbol{V}(x) = F(x)$ by a well known property of the volume potential.

At a point $x_0 \in B$, taking the cartesian coordinate system such that $x_0 = (a \ , 0 \ , 0)$, $a = \|x_0\|$ we have , integrating without difficulties :

$$\boldsymbol{V}(x_0) = \int_0^{2\pi} \int_{-R}^{R} \int_0^{\sqrt{R^2 - s^2}} \frac{G\rho r}{\sqrt{(s-a)^2 + r^2}} dr\, ds\, d\phi = 2\pi G\rho \left(R^2 - \frac{1}{3} a^2\right)$$

and so we have the well known result about the gravitational field in a homogeneous sphere

$$F(x) = -\frac{4}{3} \pi G \rho x$$

Therefore $g(r) = \frac{4}{3} \pi G \rho$ and $\gamma = \alpha g$ is constant in the interior of the sphere.

Now from equations (21) , (23) we obtain

$$\mathscr{L}_l (\mathscr{L}_l - \sigma)(\mathscr{L} - p\,\sigma) W = -l(l+1) C_l W \qquad (31)$$ where

$C_l = \frac{2\beta\gamma}{\kappa\nu} R^4$. Let $F = l(l+1) \frac{\gamma}{\nu} R^4 \Theta$. According (23) and (31) we have :

$$\mathscr{L}_l(\mathscr{L}_l - \sigma)W = F \qquad (32)$$

$$(\mathscr{L}_l - p\,\sigma)F = -l(l+1)C_l W \qquad (33)$$

$$W = F = 0 \;\;,\;\; \frac{d^2 W}{d\,r^2} = 0 \quad \text{at} \quad r = 1 \qquad (34)$$

For φ , ψ some arbitrary class C^2 functions on the [0,1] interval , integrating by parts we have :

$$\int_0^1 r^2 \varphi \mathscr{L}_l \psi\, d\,r = r^2 \varphi \frac{d\,\psi}{d\,r}\Big|_0^1 - \int_0^1 \left\{ r^2 \frac{d\,\varphi}{d\,r}\frac{d\,\psi}{d\,r} + l(l+1)\varphi\psi \right\} d\,r = \qquad (35)$$
$$= r^2\left(\varphi \frac{d\,\psi}{d\,r} - \psi \frac{d\,\varphi}{d\,r}\right)\Big|_0^1 + \int_0^1 r^2 \psi \mathscr{L}_l \varphi\, d\,r$$

Therefore, multiplying (33) by $r^2 F^*$ and integrating over r , substituting for the conjugate F^* in the obtained right side according to (32) and using (35) considering the boundary conditions (34) we obtain after calculus :

$$\int_o^1 \left\{ r^2 \left|\frac{d\,F}{d\,r}\right|^2 + l(l+1)|F|^2 + p\,\sigma r^2 |F|^2 \right\} d\,r -$$
$$- l(l+1)C_l \left\{ -2\left[r \left|\frac{d\,W}{d\,r}\right|^2 \right]_0^1 + \int_0^1 r^2 |\mathscr{L}_l W|^2 d\,r + \sigma^* \int_0^1 \left[r^2 \left|\frac{d\,W}{d\,r}\right|^2 + l(l+1)|W|^2 \right] d\,r \right\} = 0 \qquad (36)$$

Taking the imaginary part of (36) , it follows :

$$\Im\,\sigma \left\{ p \int_0^1 r^2 |F|^2 d\,r + l(l+1)C_l \int_0^1 \left[r^2 \left|\frac{d\,W}{d\,r}\right|^2 + l(l+1)|W|^2 \right] d\,r \right\} = 0$$

The factor of $\Im\,\sigma$ in this equation is positive definite and so $\Im\,\sigma = 0$
Since σ is real, there are no oscillatory modes and the change of stability, when the heat sources defined by β are changing , occurs via marginal stationary states with $\sigma = 0$, which states determine also the convection patterns.
The equations governing the marginal stationary state are :

$$\mathscr{L}_l Z = 0 \qquad (37)$$

$$\mathscr{L}_l^2 W = F \qquad (38)$$

$$\mathscr{L}_l F = -l(l+1)C_l W \qquad (39)$$

$$F = l(l+1)\frac{\gamma}{\nu} R^4 \Theta \qquad (40)$$

with boundary conditions $W = F = 0$, $\dfrac{d^2 W}{d\,r^2} = 0$ at $r = 1$ $\qquad (41)$

We have also in spherical polar coordinates :

$$r\,\omega_r = x \cdot (\nabla \times u) = \frac{\partial u_\phi}{\partial \vartheta} - \frac{1}{\sin(\vartheta)}\frac{\partial u_\phi}{\partial \phi} + \cot(\vartheta) u_\phi \qquad (42)$$

Applying $\dfrac{\partial}{\partial r} - \dfrac{1}{r}$ in (42) and considering (24) and (25) it follows that :

$$\left(\frac{d}{d\,r} - \frac{1}{r}\right) Z = 0 \quad \text{at} \; r = 1 \qquad (43)$$

It is easy to prove, solving (37), that if $l \neq 1$, (40) leads to

$Z\equiv 0$ and if $l=1$ then $Z=Cr$ with C constant

We will however consider $Z\equiv 0$ and so, according (30) u is entirely poloidal
Consider an elementary marginal stationary state (as shown above this is also an onset of instability) which corresponds to an spherical harmonic of order *l* :

$$r u_r = W(r) Y_l^m(\vartheta, \phi) \quad (44)$$

$$\theta = \Theta(r) Y_l^m(\vartheta, \phi) \quad (45)$$

We expand *F* in a Fourier-Bessel series of the form

$$F = \frac{1}{\sqrt{r}} \sum_j A_j J_{l+\frac{1}{2}}(\alpha_{l,j} r) \quad (46)$$ where $J_{l+\frac{1}{2}}$ denotes the Bessel function of order $l+\frac{1}{2}$

and $\alpha_{l,j}$ is its j-th zero and we will consider A_j as variational parameters.
Clearly *F* = 0 at *r* = 1 and so the boundary condition for *F* is satisfied .
From (38) we can express *W* in the form

$$W = \sum_j A_j W_j \quad (47)$$ where $\mathscr{L}_l^2 W_j = \frac{1}{\sqrt{r}} J_{l+\frac{1}{2}}(\alpha_{l,j} r)$.

Since $\mathscr{L}_l \left(\frac{J_{l+\frac{1}{2}}(\alpha r)}{\sqrt{r}} \right) = \frac{-\alpha^2}{\sqrt{r}} J_{l+\frac{1}{2}}(\alpha r) \quad (48)$ we have the general solution which is free of singularity at the origin :

$$W_j = \frac{1}{\alpha_{l,j}^4} \frac{J_{l+\frac{1}{2}}(\alpha_{l,j} r)}{\sqrt{r}} + B_j r^l + C_j r^{l+2} \quad (49)$$ (because the equation $\mathscr{L}_l^2 W = 0$ admits the fundamental system of solutions $(r^l , r^{l+2} , r^{-l-1} , r^{-l+1})$) where B_j , C_j are constants.

The condition $W_j = 0$ at $r=1$ requires $B_j = -C_j$
Since at $r=1$, considering also (48) we have

$$\frac{d^2 W_j}{dr^2} = \mathscr{L}_l W_j - \left(\frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} \right) W_j = -2(2l+1) B_j - \frac{2 J'_{l+\frac{1}{2}}(\alpha_{l,j})}{\alpha_{l,j}^3} ,$$

the condition $\frac{d^2 W_j}{dr^2} = 0$ at $r=1$ requires

$$B_j = -\frac{1}{2l+1} \frac{J'_{l+\frac{1}{2}}(\alpha_{l,j})}{\alpha_{l,j}^3} \quad (50)$$

Now substituting *F* and *W* in (39) according to (46) and (49) we obtain :

$$\sum_j A_j \alpha_{l,j}^2 \frac{J_{l+\frac{1}{2}}(\alpha_{l,j} r)}{\sqrt{r}} = l(l+1) C_l \sum_j A_j \left\{ \frac{1}{\alpha_{l,j}^4} \frac{J_{l+\frac{1}{2}}(\alpha_{l,j} r)}{\sqrt{r}} + B_j (r^l - r^{l+2}) \right\} \quad (51)$$

Because of the orthogonality relation :

$\int_0^1 r J_{l+\frac{1}{2}}(\alpha_{l,j} r) J_{l+\frac{1}{2}}(\alpha_{l,k} r) dr = \frac{1}{2}\delta_{jk}\left[J'_{l+\frac{1}{2}}(\alpha_{l,j})\right]^2$ for $j, k \in \mathbb{N}^*$ which is satisfied by the Bessel functions, multiplying equation (51) by $r^{\frac{3}{2}} J_{l+\frac{1}{2}}(\alpha_{l,k} r)$ and integrating over *r*, we obtain

$$\frac{1}{2}\left[J'_{l+\frac{1}{2}}(\alpha_{l,k})\right]^2 \alpha_{l,k}^2 A_k = l(l+1) C_l \left\{ \frac{1}{2} \frac{\left[J'_{l+\frac{1}{2}}(\alpha_{l,k})\right]^2}{\alpha_{l,k}^4} + \sum_j (k|j) A_j \right\} \quad (52)$$ for $k = 1, 2, 3 \ldots$

where $(k|j) = B_j \int_0^1 (r^{l+\frac{3}{2}} - r^{l+\frac{7}{2}}) J_{l+\frac{1}{2}}(\alpha_{l,k} r) dr$ (53)

The Bessel functions satisfy the relations :

$$z^{\nu+1} J_\nu(z) = \frac{d}{dz}[z^{\nu+1} J_{\nu+1}(z)]$$

$$J_{\nu+2} + J_\nu = 2\frac{\nu+1}{z} J_{\nu+1}$$

$$z J'_\nu = \nu J_\nu - z J_{\nu+1}$$

Using these relations, from (53) and (50), follows without difficulties that we have :

$$(k|j) = 2B_j \frac{J_{l+\frac{5}{2}}(\alpha_{l,k})}{\alpha_{l,k}^2} = \frac{4(l+\frac{3}{2})}{2l+1} \frac{J'_{l+\frac{1}{2}}(\alpha_{l,j}) J'_{l+\frac{1}{2}}(\alpha_{l,k})}{\alpha_{l,j}^3 \alpha_{l,k}^3} \quad (54)$$

and so $(k|j) = (j|k)$ which reflects the self-adjoint character of the characteristic value problem for C_l

The system of equations (52) must have proper solutions for A_j , and so approximations for the characteristic value C_l can be obtained by restricting to a finite number *n* of equations in (52) and requiring the corresponding determinant of the system for the unknown variables $(A_j)_{j=\overline{1,n}}$ to be zero. This gives an equation for the characteristic value C_l

With $(k|j)$ given by (54) and $D_j = \frac{J'_{l+\frac{1}{2}}(\alpha_{l,j})}{\alpha_{l,j}^3} A_j$ for $j = 1, 2, 3\ldots$ the system (52) can be rewritten in the form :

$$\sum_j \left\{ \frac{-(2l+1)}{4(2l+3)} \alpha_{l,k}^8 \left[\frac{1}{l(l+1)C_l} - \frac{1}{\alpha_{l,k}^6} \right] \delta_{jk} + 1 \right\} D_j = 0 \quad (55)$$

The required secular determinant is :

$$\left\| -\frac{2l+1}{4(2l+3)} \alpha_{l,k}^8 \left[\frac{1}{l(l+1)C_l} - \frac{1}{\alpha_{l,k}^6} \right] \delta_{jk} + 1 \right\|_{j,k} = 0$$

This equation determines C_l .

The coefficients A_j now can be determined from (52) by setting an arbitrary value on A_1 .

A first approximation to the value of C_l can be obtained by setting the (1,1) element of the secular matrix above, equal to zero.

Thus in first approximation we will have :

$$l(l+1) C_l = \frac{\alpha_{l,1}^8}{\alpha_{l,1}^2 + \frac{4(2l+3)}{2l+1}}$$

$$W=\frac{1}{\alpha_{l,1}^4}\frac{J_{l+\frac{1}{2}}(\alpha_{l,1}r)}{\sqrt{r}}+\frac{1}{2l+1}\frac{J_{l+\frac{3}{2}}(\alpha_{l,1})}{\alpha_{l,1}^3}(r^l-r^{l+2})$$

$$F=\frac{1}{\sqrt{r}}J_{l+\frac{1}{2}}(\alpha_{l,1}r)$$

$$\Theta=\frac{\nu}{\gamma}\frac{1}{l(l+1)R^4}\frac{1}{\sqrt{r}}J_{l+\frac{1}{2}}(\alpha_{l,1}r)$$

The velocity field, as we noticed is entirely poloidal and so we have :

$$\bar{S}=\frac{rW}{l(l+1)}$$

$$u_r=\frac{1}{r^2}l(l+1)\bar{S}Y_l^m(\vartheta\ ,\ \phi)$$

$$u_\vartheta=\frac{1}{r}\frac{d\bar{S}}{dr}\frac{\partial Y_l^m}{\partial\vartheta}(\vartheta\ ,\ \phi)$$

$$u_\phi=\frac{1}{r\sin(\vartheta)}\frac{d\bar{S}}{dr}\frac{\partial Y_l^m}{\partial\phi}(\vartheta\ ,\ \phi)$$

The streamlines equations are $\frac{dx_1}{u_1}=\frac{dx_2}{u_2}=\frac{dx_3}{u_3}$ or in spherical polar coordinates

$$\frac{dr}{u_r}=\frac{r\,d\vartheta}{u_\vartheta}=\frac{r\sin(\vartheta)d\phi}{u_\phi}\qquad(56)$$

For *l* = 1 , *m* = 0 we have $Y_l^m(\vartheta\ ,\ \phi)=P_1^0(\cos(\vartheta))=\cos(\vartheta)$ and the equations (56) become :

$$\frac{1}{2\bar{S}}\frac{d\bar{S}}{dr}dr=-\cot(\vartheta)d\vartheta$$

$$d\phi=0$$

Integrating these equations we obtain, in terms of *W* :

$\sin(\vartheta)=\frac{1}{\sqrt{rW}}\sin(\vartheta_0)\sqrt{r_0W(r_0)}$ where $r_0\ ,\ \vartheta_0\ ,\ \phi_0$ are the coordinates of a single point

$$\phi=\phi_0$$

belonging to the streamline (we observe that the streamline remains in the same meridional section)
Also we have the temperature distribution field

$$T=T_0+\Theta\cos(\vartheta)$$

$$T=\beta_0-\beta r^2+\frac{\nu}{2\gamma R^4}\frac{1}{\sqrt{r}}J_{\frac{3}{2}}(\alpha_{1,1}r)\cos(\vartheta)$$

with

$$\beta=\frac{\kappa\nu}{2\gamma R^4}C_l$$

$$C_l=\frac{1}{2}\frac{\alpha_{1,1}^8}{\alpha_{1,1}^2+\frac{20}{3}}$$

We observe that $T=K_0-K_1r^2+K_2\frac{1}{\sqrt{r}}J_{\frac{3}{2}}(\alpha_{1,1}r)\cos(\vartheta)$ where $K_1\ ,\ K_2$ are positive constants depending on the fluid material and the radius of the sphere and K_0 can be interpreted as a additive temperature scale constant.

The streamlines (for $l = 1$, $m = 0$ first approximation) in a meridional section on the onset of instability pattern mode are presented in fig.1 and the corresponding temperature distribution field is presented in fig.2

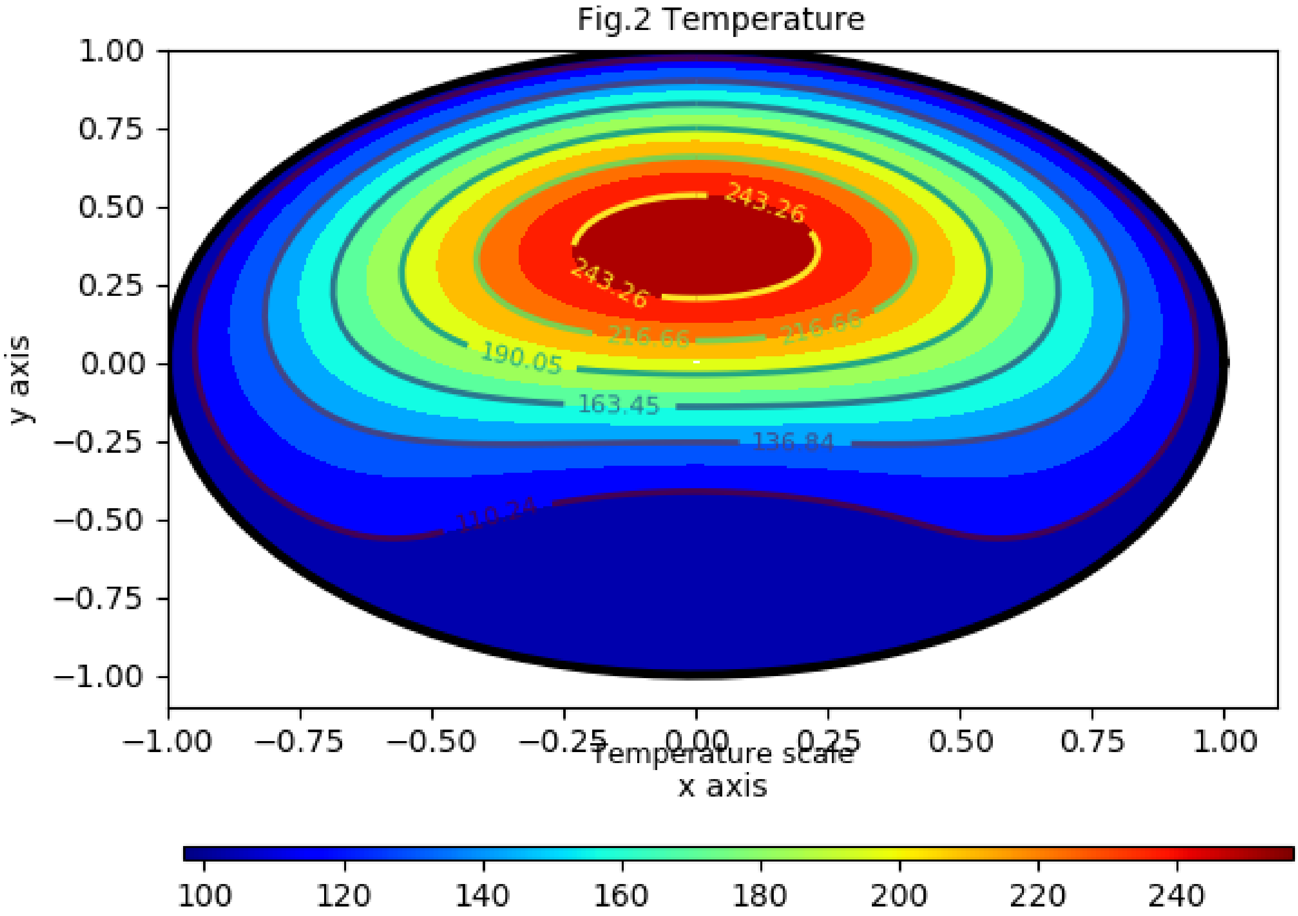
Fig.2 Temperature
y axis
x axis
Temperature scale
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.00 −0.75 −0.50 −0.25 0.00 0.25 0.50 0.75 1.00
243.26
216.66
190.05
163.45
136.84
110.24
100 120 140 160 180 200 220 240

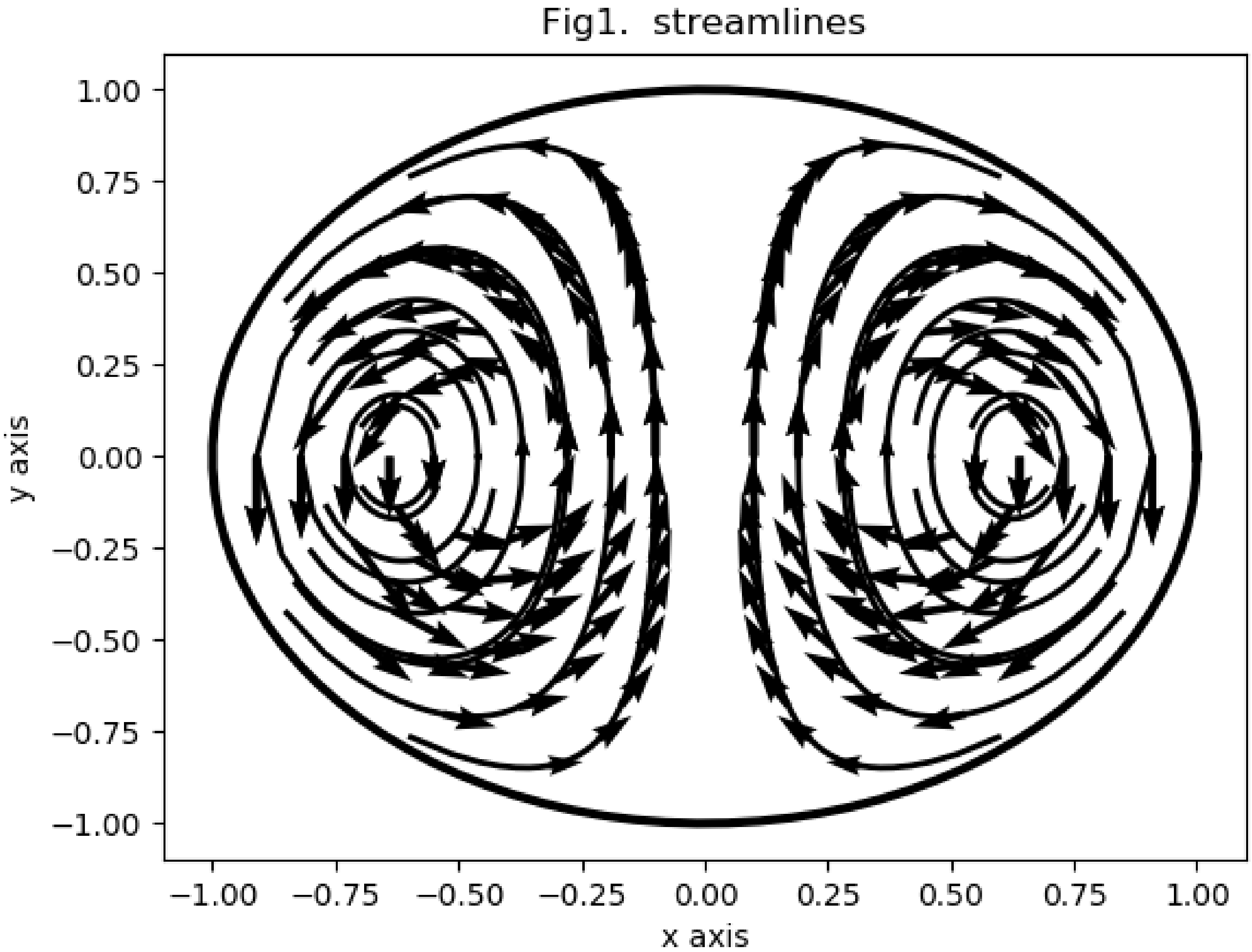
Fig1. streamlines
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y axis
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x axis

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