Toroidal poloidal decomposition

Let $u:\mathbb{R}^3\backslash\{(0\ ,\ 0\ ,\ 0)\}\rightarrow\mathbb{R}^3$ a solenoidal vector field .

Suppose first that *u* has a toroidal poloidal decomposition *u* = *T* + *P*,

$$T=\nabla\times(\Psi\bar{r}) \qquad (1)$$

$$P=\nabla\times(\nabla\times(\Phi\bar{r})) \qquad (2)$$

with $\bar{r}=(x_1\ ,\ x_2\ ,\ x_3)\in\mathbb{R}^3\ ,\ r=\|\bar{r}\|\ ,\ r\neq 0$

An alternative equivalent form of (1) and (2) is :

$$T=(\nabla\Psi)\times\bar{r} \qquad (3)$$

$$P=\nabla\times((\nabla\Phi)\times\bar{r}) \qquad (4)$$

We can verify that :

$$\bar{r}\cdot T=0$$

$$\bar{r}\cdot(\nabla\times T)=-(\bar{r}\times\nabla)\cdot(\bar{r}\times\nabla)\Psi=L^2\Psi$$

$$\bar{r}\cdot P=-(\bar{r}\times\nabla)\cdot(\bar{r}\times\nabla)\Phi=L^2\Phi$$

where $L^2=x_i\frac{\partial}{\partial x_i}+x_i\frac{\partial}{\partial x_i}x_j\frac{\partial}{\partial x_j}-r^2\nabla^2=r^2\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\nabla^2\right)$

(with the Einstein summation convention for repeating indexes)
or in spherical polar coordinates,

$x_1=r\cos(\vartheta)\ ,\ x_2=r\sin(\vartheta)\cos(\varphi)\ ,\ x_3=r\sin(\vartheta)\sin(\varphi)$

with $r>0\ ,\ \vartheta\in(0\ ,\ \pi)\ ,\ \varphi\in(0\ ,\ 2\pi)$

$$L^2=-\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}-\frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\phi^2}$$

L^2 is the spherical harmonics operator, or the negative square of angular momentum operator

Also we can verify that $\bar{r}\cdot(\nabla\times P)=0$ if Φ is a class C^3 function

Hence if $\Psi\in C^2$ and $\Phi\in C^3$ we have

$$B_r=\bar{r}\cdot u=\bar{r}\cdot P=L^2\Phi$$

$$J_r=\bar{r}\cdot(\nabla\times u)=\bar{r}\cdot(\nabla\times T)=L^2\Psi$$

Now let $B=B(r\ ,\ \Theta)$, with $\Theta\in S=\{\bar{r}\in\mathbb{R}^3\,|\,\|\bar{r}\|=1\}$, having the expansion in spherical harmonics :

$$B(r,\Theta)=\sum_{n=1}^{\infty}\sum_{k=-n}^{n}a_{nk}(r)Y_n^k(\Theta) \quad \text{with} \quad Y_n^k(\vartheta,\varphi)=P_n^{|k|}(\cos(\vartheta))e^{ik\varphi}$$

where for $k\in\mathbb{N}\ ,\ 0\leq x\leq 1\ :\ P_n^k(x)=(-1)^k(1-x^2)^{k/2}\frac{d^kP_n^0}{dx^k}(x)$

are the associated Legendre polynomials, $P_n^0=P_n$ is the Legendre polynomial of degree *n*,

$$P_n^{-k}=(-1)^k\frac{(n-k)!}{(n+k)!}P_n^k$$

For a polynomial *f* of degree n and $s\in\mathbb{N}$ we have Markov's inequality (see [1],[2]) :

$$\max_{-1\leq t\leq 1}\left|\frac{d^sf}{dt^s}(t)\right|\leq Mn^{2s} \quad \text{where} \quad M=\max_{-1\leq t\leq 1}|f(t)|$$

and so if $K_{nk}^i=\left\|D^i\,Y_n^k\right\|$, where D denotes the total differential operator with respect to $(\vartheta\ ,\ \varphi)$, we have : $K_{nk}^i\leq 72n^{2|k|+2i}$ for $i=0,1,2,3$

(the 72 coefficient comes from the multiple derivation terms)

Therefore, by a well known series/sequences differentiation theorem , if

a_{nk} are class C^3 functions and satisfy $\sum_{n=1}^{\infty}\sum_{k=-n}^{n} n^{2|k|+4}\left|a_{nk}^{(i)}\right| \leq \infty$ for $i=0,1,2,3$
uniformly for r on compact sets in the definition domain, then the function defined as :
$\Phi(r,\Theta)=\sum_{n=1}^{\infty}\sum_{k=-n}^{n}\frac{1}{n(n+1)}a_{nk}(r)Y_n^k(\Theta)$ is class C^3 and satisfies $L^2\Phi=B$
since as we know $L^2 Y_n^k=n(n+1)Y_n^k$

Returning to the solenoidal class C^1 vector field u we suppose now that
$\bar{r}\cdot u=B_r=\sum_{n=1}^{\infty}\sum_{k=-n}^{n}a_{nk}(r)Y_n^k(\Theta)$ is with a_{nk} class C^3 functions satisfying

$$\sum_{n=1}^{\infty}\sum_{k=-n}^{n} n^{2|k|+4}\left|a_{nk}^{(i)}\right| \leq \infty \quad \text{for } i=0,1,2,3 \qquad (5)$$

$\bar{r}\cdot(\nabla\times u)=J_r=\sum_{n=1}^{\infty}\sum_{k=-n}^{n}c_{nk}(r)Y_n^k(\Theta)$ is with c_{nk} class C^2 functions satisfying

$$\sum_{n=1}^{\infty}\sum_{k=-n}^{n} n^{2|k|+2}\left|c_{nk}^{(i)}\right| \leq \infty \text{ for } i=0,1,2 \qquad (6)$$

(5) and (6) must be valid for r on compact sets in the definition domain.
Under these circumstances we can define
$\Phi=\sum_{n=1}^{\infty}\sum_{k=-n}^{n}\frac{1}{n(n+1)}a_{nk}(r)Y_n^k(\Theta)$ and $\Psi=\sum_{n=1}^{\infty}\sum_{k=-n}^{n}\frac{1}{n(n+1)}c_{nk}(r)Y_n^k(\Theta)$
Φ is class C^3 and Ψ is class C^2 and also

$L^2\Phi=B_r$ and $L^2\Psi=J_r$
Let $P=\nabla\times(\nabla\times(\Phi\bar{r}))$ and $T=\nabla\times(\Psi\bar{r})$ (*)
From the considerations above follows that T , P are class C^1 functions and taking
$A=u-T-P$, we have that A is class C^1 and satisfies :
$\bar{r}\cdot A=0$, $\nabla\cdot A=0$, $\bar{r}\cdot(\nabla\times A)=0$ or in spherical polar coordinates :
$A_r=0$

$$A_{\vartheta,\vartheta}+\frac{1}{\sin(\vartheta)}A_{\varphi,\varphi}+\cot(\vartheta)A_\vartheta=0$$

$$A_{\varphi,\vartheta}-\frac{1}{\sin(\vartheta)}A_{\vartheta,\varphi}+\cot(\vartheta)A_\varphi=0$$

For $U=A_\vartheta+A_\varphi$ and $V=A_\vartheta-A_\varphi$ we have :
$(U\sin(\vartheta))_{,\vartheta}-V_{,\varphi}=0 \qquad (7)$
$(V\sin(\vartheta))_{,\vartheta}+U_{,\varphi}=0 \qquad (8)$
and so exists a class C^2 potential $W=W(\vartheta,\varphi)$ periodic with period 2π in ϑ and φ
such that $W(2\pi-\vartheta,\varphi)=-W(\vartheta,\pi+\varphi)$ (that corresponds to the mapping of
$[0,2\pi]\times[0,2\pi]$ on the unit sphere) and
$W_{,\varphi}=U\sin(\vartheta)$, $W_{,\vartheta}=V \qquad (9)$
Relations (8) and (9) lead to :
$W_{,\vartheta\vartheta}\sin^2(\vartheta)+W_{,\vartheta}\sin(\vartheta)\cos(\vartheta)+W_{,\varphi\varphi}=0 \qquad (10)$

Expanding W in Fourier series ,

$W=\sum_{n=-\infty}^{\infty} W_n(\vartheta)e^{in\varphi}$ we have from (10) respective (9) that for $n\neq 0$

$$W_{n,\vartheta\vartheta}\sin^2(\vartheta)+W_{n,\vartheta}\sin(\vartheta)\cos(\vartheta)-n^2W_n=0 \text{ and } W_n(0)=W_n(\pi)=0 \quad (11)$$

Let $n\neq 0$ be given and $H=\Re W_n$ (the same will be done for $H=\Im W_n$)

Then from (11) we have

$$(1-\cos(2\vartheta))H_{,\vartheta\vartheta}+H_{,\vartheta}\sin(2\vartheta)=2n^2H \text{ and } H(0)=H(\pi)=0 \quad (12)$$

Multiplying (12) by H and integrating over $\vartheta\in[0,\pi]$, after integrating several times by parts, we have :

$$-2\int_0^{\pi} H_{,\vartheta}^2\sin^2(\vartheta)d\vartheta=\int_0^{\pi}(2n^2-\cos(2\vartheta))H^2d\vartheta$$

The integrand on the right side of this equality is positive and the left side integrand is negative. Therefore *H* must be zero for $n\neq 0$ and so *U* and *V* not depend on φ

Hence from (7) and (8) follows $U=\frac{C_1}{\sin(\vartheta)}$ and $V=\frac{C_2}{\sin(\vartheta)}$

with C_1 , C_2 constants . But U , V must be bounded on $[0,\pi]$ and so A is identical 0

Thus u has the toroidal poloidal decomposition $u=T+P$

with T , P determined by $(*)$

REFERENCES

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