Special relativity, Lorentz transformations

Consider the affine euclidean space of space-time events in special relativity, U
A reference frame will be identified with a bijective function $R: U \to \mathbb{R}^4$
For a space-time event $P \in U$ we have $R(P) = (\bar{x}, t)$ with $\bar{x} = (x_1, x_2, x_3)$ spatial coordinates and t time coordinate.
$R^{-1}(\bar{0}) = O$ with $\bar{0} = (0,0,0,0)$ and $\overline{OP} = (\bar{x}, t)$
For two reference frames *R* , *R'* we have the coordinate transformations
$T: \mathbb{R}^4 \to \mathbb{R}^4$, $T = R' \circ R^{-1}$, $R(O) = \bar{0}$, $R'(O') = \bar{0}$
$T(\overline{OP}) = \overline{O'P'} = T(\bar{0}) + M(\overline{OP})$
$T(\bar{0}) = \overline{O'O}$, $M(\bar{0}) = \bar{0}$
We consider now $\mathbb{R}^4 = E$ as the four dimensional euclidean space of points which has at the same time the four dimensional real vector space structure.
Suppose that *R* , *R'* are inertial reference frames. The fact that if a particle is moving uniformly rectilinear as it is seen in the frame *R* then it will be seen moving uniformly rectilinear in the frame *R'* leads to the fact that *T* transforms any straight line of *E* into a straight line :
for any points $A, B \in E$ we have $T(AB) = T(A)T(B)$
If $(A_1 A_2 A_3)$ is a plane in E and $P \in (A_1 A_2 A_3)$ we can take, after eventually renumbering, a point $Q = PA_1 \cap A_2 A_3$. Therefore we will have
$Q' = T(Q) = T(PA_1) \cap T(A_1 A_2) = P'A'_1 \cap A'_2 A'_3$ and so $P' = T(P) \in (A'_1 A'_2 A'_3)$
T transforms any plane into a plane.
Thus if $AB \| CD$, $A, B, C, D \in E$ then A, B, C, D are coplanar and so *A',B',C',D'* are also coplanar and $A'B' \cap C'D' = T(AB) \cap T(CD) = \emptyset$ which means $A'B' \| C'D'$
On $E = \mathbb{R}^4$ we have the compatible affine structure given by $\overline{AB} = B - A$ for any $A, B \in E$
For any $A, B \in E$, considering $S = \bar{0}$ and the parallelogram $[SACB]$ we will have also $[S'A'C'B']$ as a parallelogram and so

$$C = A + B$$

$T(C) = T(A) + T(B) - T(S)$ which easily leads to

$$M(A+B) = M(A) + M(B)$$

For *A, B, C, D* in *E* such that *C* is the middle of the segment *AB* and *B* is the middle of the segment *AD*, considering the parallelograms [*AGBF*] and [*AKDH*] where *B* is the middle of the segment *KH* we will have that [*A'G'B'F'*] and [*A'K'D'H'*] are also parallelograms with centres at *C'* respective *B'* and so *C'* is the middle of segment *A'B'* and *B'* is the middle of the segment *A'D'*.
Therefore it is easy to prove by induction that

any $A \in E = \mathbb{R}^4$, $m, n \in \mathbb{Z}$ satisfy $M(\frac{m}{2^n} A) = \frac{m}{2^n} M(A)$

<u>Lemma</u>

$\{\frac{m}{2^n} \in \mathbb{Q} | m \in \mathbb{Z}, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$

Proof :

For $\alpha \in \mathbb{R}$, $\epsilon \in \mathbb{R}_+$
we can take $p, m \in \mathbb{Z}$, $q, n \in \mathbb{N}^*$ such that
$\frac{1}{q} < \frac{\epsilon}{2}$, $\left|\frac{p}{q} - \alpha\right| < \frac{\epsilon}{2}$, $\frac{1}{2^n} < \frac{1}{2q}$, $\frac{m}{2^n} \in \left(\frac{p}{q}, \frac{p+1}{q}\right)$

and so $\left|\frac{m}{2^n} - \alpha\right| < \frac{1}{q} + \left|\frac{p}{q} - \alpha\right| < \epsilon$

If *T* is continuous it follows now that $M(\alpha A) = \alpha M(A)$ for any $A \in \mathbb{R}^4$, $\alpha \in \mathbb{R}$

Hence, under the considered assumptions, *M* is linear continuous.
$T(u)=T(\bar{0})+M(u)$ for $u\in\mathbb{R}^4$

For $(\bar{x},t)=(x_1,x_2,x_3,x_4)$ and $(\bar{x}',t')=(x'_1,x'_2,x'_3,x'_4)$ we have
$x'_k=M_{kl}x_l$ (with Einstein summation convention) , or taking $(\bar{x},t)$ and $(\bar{x}',t')$ as column vectors *X* respective *X'*, we write *X' = MX*.
Let $G=(\gamma_{pq})$ with $\gamma_{\alpha\beta}=\delta_{\alpha\beta}$, $\gamma_{4\alpha}=\gamma_{\alpha 4}=0$, $\gamma_{44}=-c^2$ for $\alpha,\beta=1,2,3$

(where *c* is the speed of light constant)
Considering $O,P\in U$ separated by a light signal, for $\overline{OP}=X^T$ we have $X^T G X=0$ because the light signal travels from *O* to *P* with the speed *c*.
We assume, according to special relativity, that the speed of light is the same constant in any inertial reference frame and so it follows that if $X^T\in\mathbb{R}^4$ satisfies $X^T G X=0$ then $X'^T G X'=0$
Therefore if $X^T\in\mathbb{R}^4$ and $X^T G X=0$ then $X^T M^T G M X=0$ (1)
Taking $X^T=(\pm ct,0,0,t)$, $M^T G M=(a_{kl})$ it follows
$a_{11}c^2t^2+a_{44}t^2\pm 2a_{14}ct^2=0$ for any $t\in\mathbb{R}$ and so $a_{14}=a_{41}=0$,$-a_{44}=c^2a_{11}$
In the same way we have $a_{44}=c^2a_{\alpha\alpha}$, $a_{\alpha 4}=a_{4\alpha}=0$ for $\alpha=1,2,3$
(the matrix M^TGM being obviously symmetric)
Taking $X^T=(cx_1,\pm cx_2,0,\sqrt{x_1^2+x_2^2})$ it follows now
$a_{44}x_1^2+a_{44}x_2^2-a_{44}(x_1^2+x_2^2)\pm 2a_{12}x_1x_2=0$ for any $x_1,x_2\in\mathbb{R}$ and so $a_{12}=0$
In the same way we obtain $a_{\alpha\beta}=0$ for $\alpha\neq\beta$
Hence exists $l\in\mathbb{R}$ such that $lG=M^TGM$
We suppose now that the frame R moves with constant velocity $\bar{v}'$ in R'
(i.e. a point at rest in R moves with velocity $\bar{v}'$ in R')
and R' moves with constant velocity $\bar{v}$ in R
We take $S:\mathbb{R}^4\rightarrow\mathbb{R}^4$, $S=T-T(\bar{0})$ and we have $S\circ R'\circ R^{-1}=M$
Let $Ort=\{(a_{kl})\in M_{4\times 4}(\mathbb{R})|a_{\alpha 4}=a_{4\alpha}=0\ ,\ (a_{\alpha\beta})=Q\ ,\ \alpha,\beta=1,2,3\ ,\ Q^TQ=\mathbf{I}\ ,\ a_{44}=1\}$

Rotating adequately the frames $S\circ R'$ and R we can find $Q,P\in Ort$ such that $P\circ R=R_1$ moves with velocity $(0,0,v')$, $v'=\|\bar{v}'\|$ in $Q\circ S\circ R'=R_0$ and R_0 moves with velocity $(0,0,v)$, $v=-\|\bar{v}\|$ in R_1
Obviously we have $R_0\circ R_1^{-1}=QMP^T=\bar{M}$, $lG=\bar{M}^TG\bar{M}$ (2)

It follows that for $\bar{M}=(m_{kl})$ and $\bar{M}^{-1}=(m^*_{kl})$ we have
$(m_{k4})_k=(0,0,sv',s)$, $s\neq 0$
$(m^*_{k4})_k=(0,0,s^*v,s^*)$, $s^*\neq 0$
Let $N=(\sqrt{\gamma_{pq}})$ (with $\sqrt{-c^2}=ic$, $p,q=\overline{1,4}$) , $Z=N\bar{M}N^{-1}$ (5)
From (2) follows $l\mathbf{I}=Z^TZ$, $lZ^{-1}=Z^T$ (6)

By calculus from (5) , (3) , (4) we have

$$Z=\begin{pmatrix} . & . & . & 0 \\ . & . & . & 0 \\ . & . & . & -isv'/c \\ . & . & . & s \end{pmatrix}, \; Z^{-1}=\begin{pmatrix} . & . & . & 0 \\ . & . & . & 0 \\ . & . & . & -is^{*}v/c \\ . & . & . & s^{*} \end{pmatrix}$$

Hence , considering (6) we will have

$$Z=\begin{pmatrix} . & . & . & 0 \\ . & . & . & 0 \\ . & . & . & -isv'/c \\ . & . & -is^{*}lv/c & s \end{pmatrix}, \; s=s^{*}l \; , \; l=-s^{*2}l^{2}v^{2}/c^{2}+s^{2}=-s^{2}v'^{2}/c^{2}+s^{2}$$

Therefore $v=-v'$ (because we have chosen $v'>0$ and $v<0$)

Since $v^2<c^2$ we have $l>0$
By calculus, it follows

$$\bar{M}=N^{-1}ZN=\begin{pmatrix} . & . & . & 0 \\ . & . & . & 0 \\ . & . & . & sv' \\ . & . & -sv/c^{2} & s \end{pmatrix}, \; \bar{M}^{-1}=N^{-1}Z^{-1}N=\begin{pmatrix} . & . & . & 0 \\ . & . & . & 0 \\ . & . & . & s^{*}v \\ . & . & -s^{*}v'/c^{2} & s^{*} \end{pmatrix}$$

$$t=s'\left(t'-\frac{v'}{c^2}x'_3\right), \; t'=s\left(t-\frac{v}{c^2}x_3\right), \; x'_3=s(x_3-vt)$$

We admit the causality principle so that if an event precedes an other event at the same spatial point as it is seen in the inertial reference frame R_1 then that one event precedes the other one event also in the inertial reference frame R_0 . Therefore we have

$$s>0 \; , \; s=\sqrt{l}\,\beta \text{ where } \beta=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Since we have $l>0$ such that $lG=M^{T}GM$ if we consider $|\det M|=1$ we will have

$l^4=(\det M)^2$, $l=1$ and so M leaves invariant the symmetric bilinear form product on the Minkowski space $V=\mathbb{R}^4$ defined by $X\cdot Y=X^{T}GY$ with X,Y as column vectors in $\mathbb{R}^4$

Consider now $M\in M_{4\times4}(\mathbb{R})$ such that $G=M^{T}GM$. It is obbvious that also $G=M^{-T}GM^{-1}$ and $|\det M|$ = 1 and so if linearly independent vectors
$(E_k)k=\overline{1,4}$ are a Minkowski base in V (i.e.$E_p\cdot E_q=\gamma_{pq}$ for $p,q=\overline{1,4}$) then (E'_k)
with $E'_k=m^{*}_{lk}E_l$, $(m^{*}_{kl})=M^{-1}$ for $k,l=\overline{1,4}$ is also a Minkowski base of linearly independent vectors.
Obviously, if $x_kE_k=x'_kE'_k$ then $x'_k=m_{kl}x_l$

$$m_{pk}m_{ql}\gamma_{pq}=\gamma_{kl} \quad (7)$$

$m^*_{pk} m^*_{ql} \gamma_{pq} = \gamma_{kl}$ (8)

We have $m^*_{\alpha 4} m^*_{\alpha 4} - c^2 m^{*2}_{44} = -c^2$ and so $m^{*2}_{44} \geq 1$ (9)

The assumed causality principle means

$m^*_{44} > 0$. Hence we can take $v \in \mathbb{R}$, $v^2 < c^2$ such that $m^*_{44} = \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

If $v = 0$, from (8) follows $m^*_{\alpha 4} = 0$ and now for any $((x'_\alpha), t') \in \mathbb{R}^4$ we have

$m^*_{\alpha\gamma} m^*_{\alpha\epsilon} x'_\gamma x'_\epsilon - c^2 (m^*_{4\alpha} x'_\alpha + t')^2 = x'_\alpha x'_\alpha - c^2 t'^2$

This leads to $m^*_{4\alpha} = 0$ and $m^*_{\alpha\gamma} m^*_{\alpha\epsilon} = \delta_{\gamma\epsilon}$ and finally to M^{-1} , $M \in Ort$

We suppose now $v \neq 0$ and we can take

$A = \frac{1}{v\beta} m^*_{\alpha 4} E_\alpha$ with $\alpha = 1, 2, 3$

$E''_3 = \beta A + \frac{v\beta}{c^2} E_4$ (10)

It is easy to prove that we have $E'_4 = v\beta A + \beta E_4$ (11)

and $A \cdot E_4 = 0$, $A \cdot A = 1$, $E''_3 \cdot E'_4 = 0$, $E''_3 \cdot E''_3 = 1$ (12)

Consider the following system in unknown variable $X \in V$

$E'_4 \cdot X = 0$ (13)

$E_4 \cdot X = 0$ (14)

$A \cdot X = 0$ (15)

$E''_3 \cdot X = 0$ (16)

Because of (10) , (11) the system is satisfied if and only if (14) and (15) are satisfied.

Obviously we can take $E_k = (\delta_{kl})_l$ for $l, k = \overline{1, 4}$ and so it is easy to prove that we can take (E''_i) such that $E''_i \cdot E''_j = \delta_{ij}$ and for $X = E''_i$ are satisfied (13)-(14) for $i, j = 1, 2$

Therefore from (14) , (15) , (12) follows that (E''_1 , E''_2 , A) is an orthonormal basis of [E_1 , E_2 , E_3] and from (13) , (16) , (12) follows that (E''_1, E''_2 , E''_3) is an orthonormal basis of [E'_1 , E'_2, E'_3].

We can transform the Minkowski base (E_1 , E_2 , E_3 , E_4) to Minkowski base (E''_1, E''_2 , A , E_4) by orthonormal coordinate transformation $Q \in Ort$.

However we can choose E''_1 , E''_2 such that det$Q$ = 1 (if not we take -E''_1 instead of E''_1)

We can transform the Minkowski base (E''_1. E''_2 , A , E) to Minkowski base (E''_1 , E''_2 , E''_3 , E'_4) by the boost coordinate transformation

$$M_v = \begin{pmatrix} 1 & O & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta & -v\beta \\ 0 & 0 & -v\beta/c^2 & \beta \end{pmatrix}$$

We can transform the Minkowski base (E''_1 , E''_2 , E''_3 , E'_4) to Minkowski base (E'_1 , E'_2 , E'_3 ,E'_4) by orthonormal coordinate transformation $P \in Ort$

Therefore we have $M = P M_v Q$ with $P, Q \in Ort, \det Q = 1$

If $\det M = 1$ we will have $\det P = \det Q = 1$

From the relation $M = P M_v Q$ follows without difficulties that we have :

$$M_{\alpha\gamma}=\bar{R}_{\alpha\gamma}+\frac{\beta-1}{v^2}\bar{R}_{\alpha\epsilon}v_\epsilon v_\gamma \text{ where } \bar{R}_{\alpha\gamma}=P_{\alpha\epsilon}Q_{\epsilon\gamma}$$

$$M_{4\gamma}=-\frac{\beta}{c^2}v_\gamma \text{ , } M_{\gamma 4}=-\beta\bar{R}_{\gamma\epsilon}v_\epsilon \text{ , } M_{44}=\beta \qquad (17)$$

with $v_\alpha=Q_{3\alpha}v$, $v_\alpha v_\alpha=v^2$, $\beta=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, $v^2<c^2$

for $\alpha,\gamma,\epsilon=1,2,3$

and so *M* is described by 6 parameters (3 parameters for the rotation $\bar{R}$ and 3 parameters for $\bar{v}=(v_\alpha)$)

Thus we have proven that

$$\mathscr{L}=\{M\in M_{4\times 4}(\mathbb{R}) | \exists P,Q\in Ort \text{ , } v\in\mathbb{R} \text{ such that } \det P=\det Q=1 \text{ , } v^2<c^2 \text{ , } M=PM_vQ\}=$$
$$=\{M\in M_{4\times 4}(\mathbb{R}) | m^*_{44}>0 \text{ , } G=M^T GM \text{ , } \det M=1\}$$

We notice that if $M\in\mathscr{L}$ then according to (17) $m_{44}=\beta=m^*_{44}>0$ and if $M\in\{M\in M_{4\times 4}(\mathbb{R}) | m_{44}>0 \text{ , } G=M^T GM \text{ , } \det M=1\}$ then $M^{-1}\in\mathscr{L}$ and from the above we can deduce $m^*_{44}=m_{44}>0$ and so we have also

$$\mathscr{L}=\{M\in M_{4\times 4}(\mathbb{R}) | m_{44}>0 \text{ , } G=M^T GM \text{ , } \det M=1\}$$

If $M,M'\in\mathscr{L}$ we have for $M''=M'M$ calculating according to (17)

$$M''_{44}=\beta'\beta\left(\frac{\bar{v}'\cdot\bar{R}\bar{v}}{c^2}+1\right)$$

We have $\bar{v}'\cdot\bar{R}\bar{v}>-|vv'|>-c^2$ and so $M''_{44}>0$

Also $M_v^{-1}=M_{-v}$

Therefore , from the above follows that $\mathscr{L}$ is a connected 6 – dimensional Lie group, $SO^+(3,1)$ the restricted Lorentz group. The more general set of transformations that also includes 4 – dimensional translations in space-time is known as the Poincare group.