Boomerang model simulation

We present a simulation model for throwing a boomerang. The boomerang consists of three at 120° angular distance, radial disposed profiled wings. For calculating the moment of inertia tensor we will approximate each wing as an 2a x 2b sized rectangle which is placed with an axis of symmetry along the radius,and with centre at distance r_0 from a central point C, which is also the mass centre of the whole boomerang. We have therefore a body tied Cartesian reference frame $R(C,\vec{i},\vec{j},\vec{k})$ with origin at C and axis versors $\vec{i},\vec{j},\vec{k}$ such that the first wing is on $\vec{j}$ axis
Each rectangular wing has by approximation thickness 2h and mass density rm.
Therefore the tensor moment of inertia with respect to the frame $R(C,\vec{i},\vec{j},\vec{k})$ will be

$\boldsymbol{I}=\boldsymbol{I_0}+T\boldsymbol{I_0}T^T+T^2\boldsymbol{I_0}T^{2T}$ where T is the 120° rotation matrix in the $(\vec{i},\vec{j})$ plane and

$$\boldsymbol{I_0}=\frac{8}{3}rm\,h\,a\,b\begin{pmatrix} a^2+3r_0^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & a^2+b^2+3r_0^2 \end{pmatrix}$$

where in the matrix we neglected the terms in h^2.
We consider a fix Cartesian reference frame $R_1(O,\vec{i}_1,\vec{j}_1,\vec{k}_1)$
The motion of the boomerang is described by the coordinates and velocities of the mass centre in frame $R_1:(\boldsymbol{x},\dot{\boldsymbol{x}})=(\xi,\eta,\zeta,\dot{\xi},\dot{\eta},\dot{\zeta})$ where $\dot{z}$ denotes the time derivative of function z
and the Euler angles and angular velocities of mobile reference frame R in the fix frame R_1

$$(w,\dot{w})=(\psi,\theta,\varphi,\dot{\psi},\dot{\theta},\dot{\varphi})$$

For

$$m_1=\begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix},\ m_2=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix},$$

$m_3=\begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we will have $\begin{pmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{pmatrix}=m_3m_2m_1\begin{pmatrix} \bar{i}_1 \\ \bar{j}_1 \\ \bar{k}_1 \end{pmatrix}$ in matrix multiplication formalism.

The coordinates of the angular velocity vector $\boldsymbol{\omega}$ in the frame R are (see refernce [1])

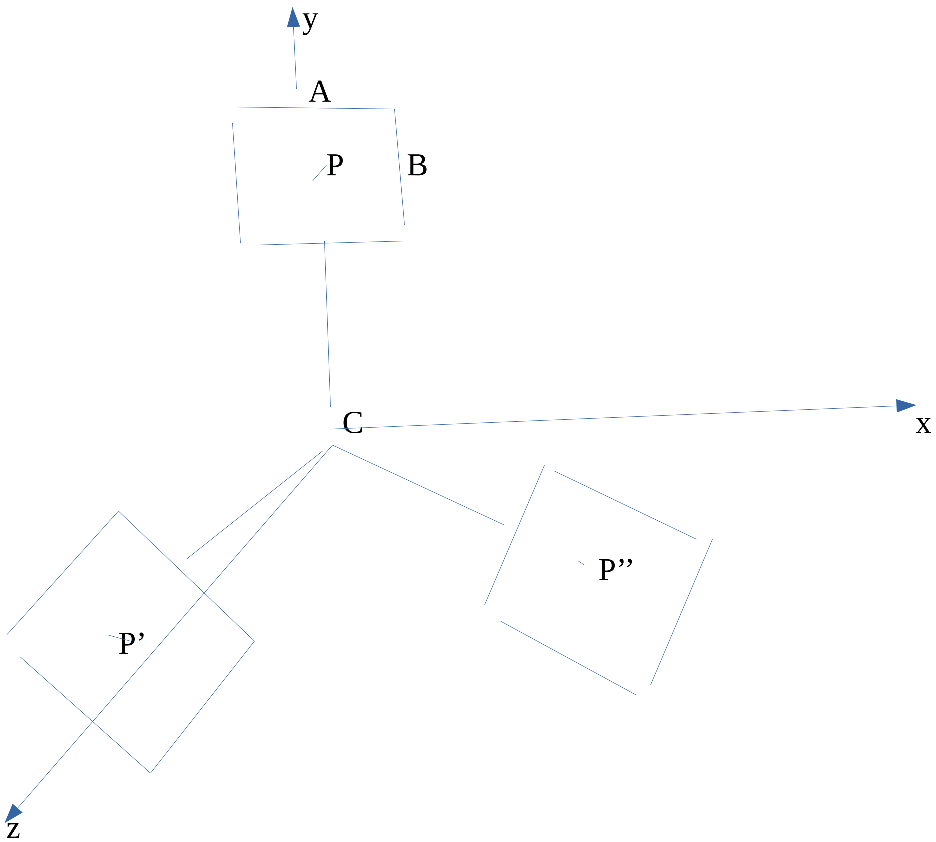
$$\omega_1=\dot{\psi}\sin(\theta)\sin(\varphi)+\dot{\theta}\cos(\varphi)$$
$$\omega_2=\dot{\psi}\sin(\theta)\cos(\varphi)-\dot{\theta}\sin(\varphi)$$

$\omega_3=\dot{\psi}\cos(\theta)+\dot{\varphi}$ and we have

$\dot{\vec{i}}=\boldsymbol{\omega}\times\vec{i}$, $\dot{\vec{j}}=\boldsymbol{\omega}\times\vec{j}$, $\dot{\vec{k}}=\boldsymbol{\omega}\times\vec{k}$ and

the motion equations for the boomerang

$\boldsymbol{I}\dot{\boldsymbol{\omega}}=M_C$ and $m_0\ddot{\boldsymbol{x}}=R$ where M_C is the resultant moment of forces with respect to C and R is the resultant of forces which are acting on the boomerang of whole mass m_0.
To determine M_C and R for each wing ,we must consider the pressure on the surface of the wing caused by the flow of air along the moving boomerang.
We hold a right handed boomerang with the right hand such that the wings plane is inclined slightly of a vertical plane and we throw it forwards at the same moment giving him a fast forwards rotation. The profiled or shaped side of the wing must show to the left of the at most 15° to the horizontal inclined initial trajectory.
During the flight, the upper rotating wing (and because of the three wings format, there is always one) rotates against the direction of translational moving and experiences a faster air flow around it and so by aerodynamic laws will have a higher resultant force acting on it which will be oriented to the left of the translational trajectory. Therefore the resultant moment from all three wings, considering the gyroscopic effect, rotates the plane of rotation to the left of the trajectory and the total resultant force direction, which is approximatively perpendicular to the wings plane will be oriented to the positive curvature of the trajectory (in this case the left of the trajectory) causing, for suitable initial conditions of throwing, a almost returning curved trajectory. Also, the component of the resultant force parallel to the wings plane will be slowing down the rotation, causing finally a decreasing of the resulting lift and the descending by gravitation course of the trajectory.



$\|CP\|=\|CP'\|=\|CP''\|=r_0$, $\|PA\|=a$, $\|PB\|=b$

$\widehat{P'CP}=\widehat{P''CP'}=\widehat{PCP''}=$ 120°

The wings plane, which contains the three rectangles with centres at P, P' and respective P'' is the xCy plane ; Cx , Cy , Cz are respective the i , j , k axes.
We can approximate, considering the throwing conditions and the shape of the wings that the aerodynamic forces acting on a wing q (q = 0,1,2) are produced in the instantaneous body tied reference frame $R(C,\bar{i},\bar{j},\bar{k})$ by a local in time stabilised stationary flow with velocity, density , pressure field $(\boldsymbol{v}_1,\rho_1,p_1)$ which is small perturbed from the field $(\boldsymbol{w}_0+\boldsymbol{v}_0,\rho_0,p_0)$

p_0,ρ_0 are constant and $\boldsymbol{w}_0+\boldsymbol{v}_0=-\omega\times r_0 T^q\bar{j}-m\dot{\boldsymbol{x}}$

where $-\boldsymbol{w}_0=\omega\times r_0 T^q \bar{j}$ is the velocity of the centre of the wing from rigid

rotation and $-\boldsymbol{v}_0=m\dot{\boldsymbol{x}}$ is the velocity of the mass centre C as they are considered with $R(C,\bar{i},\bar{j},\bar{k})$ coordinates
Moreover, the *k* component of the unperturbed flow velocity field is relatively small and will be considered not significant in producing the total aerodynamic force, considering also the aerodynamic shape of the wing.
We have
$\boldsymbol{v}_1=\boldsymbol{w}_0+\boldsymbol{v}_0+\boldsymbol{v}$, $p_1=p_0+p$, $\rho_1=\rho_0+\rho$
$(\boldsymbol{v},\rho,p)=(\boldsymbol{v},\rho,p)(x)$ is considered small of first order, depending on spatial coordinates x in mobile body tied frame *R* and in first order approximation we have the aerodynamic perturbation equations and boundary conditions :

$$\rho_0(\boldsymbol{w}_0+\boldsymbol{v}_0)\cdot\text{grad}\,\boldsymbol{v}=-\text{grad}\,p \qquad (1)$$

$$(\boldsymbol{w}_0+\boldsymbol{v}_0)\cdot\text{grad}\,\rho+\rho_0\,\text{div}\,\boldsymbol{v}=0 \qquad (2)$$

$$\text{grad}\,\rho=\frac{1}{c^2}\,\text{grad}\,p \qquad (3)$$

$$\lim_{\|x\|\to\infty}(\boldsymbol{v},\rho,p)=0 \qquad (4)$$

$$(\boldsymbol{w}_0+\boldsymbol{v}_0+\boldsymbol{v})\cdot\bar{n}=0 \text{ on } \partial D \qquad (5)$$

(1) derives from the fluid motion equations, (2) from the continuity equation , (3) contains the thermodynamic of the fluid, *c* being the speed of sound in air, (4) can be valid for a subsonic flow (we consider $(\boldsymbol{w_0}+\boldsymbol{v_0})^2 < c^2$) and $D\subset\mathbb{R}^3$ is the domain of the wing in $R(C,\bar{i},\bar{j},\bar{k})$ with $\bar{n}$ the normal on ∂D (Note that the domain depends on what wing we consider)
Hence we can apply the thin profiles theory which we present briefly below according reference [2].

<u>Thin profiles theory</u>

The linearised equations of stationary aerodynamics for a small perturbation field $(\boldsymbol{v},\rho,p)$ depending on spatial coordinates $x=(x_1,x_2,x_3)$ (see [2]) are :

$$\rho_0 V_0\boldsymbol{v}_{,1}=-\text{grad}\,p \qquad (6)$$

$$V_0 p_{,1}+c^2\,\text{div}\,\boldsymbol{v}=0 \qquad (7)$$

$$\text{grad}\,\rho=\frac{1}{c^2}\,\text{grad}\,p \qquad (8)$$

where $(V_0\bar{i},\rho_0,p_0)$ (with V_0,ρ_0,p_0 real positive constants and $\bar{i},\bar{j},\bar{k}$ the versors of the Cartesian coordinate system $x=x_1\bar{i}+x_2\bar{j}+x_3\bar{k}$) is the unperturbed state.
If the flow is around a body represented by the domain $D\subset\mathbb{R}^3$ we must have :

$$(V_0\bar{i}+\boldsymbol{v})\cdot\bar{n}=0 \text{ on } \partial D \qquad (9)(\text{ with } \bar{n} \text{ the normal on } \partial D)$$

Also we must require $\lim_{x_1\to-\infty}((\boldsymbol{v},\rho,p),\text{grad}(\boldsymbol{v},\rho,p))=0 \qquad (10)$

From (6) follows $(\text{rot}\,\boldsymbol{v})_{,1}=0$ and so $\text{rot}\,\boldsymbol{v}=F(x_2,x_3)$
Because of (10) we will have now $F=0$ and therefore the flow is irrotational and we can consider the potential $\phi=\phi(x)$ such that $\boldsymbol{v}=\text{grad}\,\phi$
From (6) and (10) we can conclude now that $p=-\rho_0 V_0 v_1$ where $\boldsymbol{v}=(v_1,v_2,v_3)$ and from (7) follows $(1-M_0^2)\phi_{,11}+\phi_{,22}+\phi_{,33}=0$ where $M_0^2=\dfrac{V_0^2}{c^2}$ is the Mach number of the unperturbed flow.

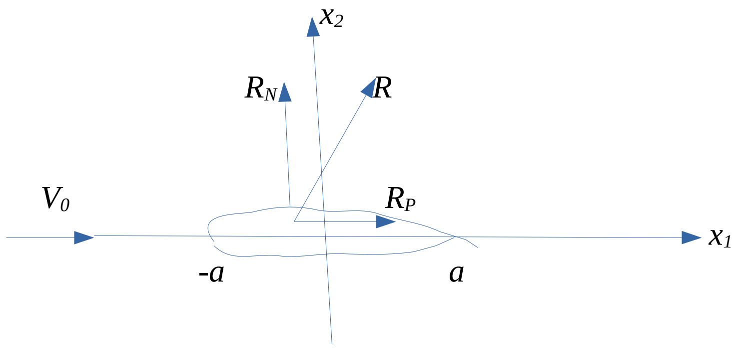
For the bidimensional case of a thin identical profiled and potential infinite in the x_3 direction wing we can reduce the x_3 dependence and consider the domain of the wing as

$D=\{(x_1,x_2)\in\mathbb{R}^2 | -a\leq x_1\leq a$ and $h_-(x_1)\leq x_2\leq h_+(x_1)\}$

with $a>0$ and with $|h_\pm|,|h'_\pm|$ small of first order

Also for a subsonic flow (Mach number less than 1) we assume that $h'_\pm$ is Hölderian what means that we have $A,\alpha>0$, $\alpha<1$ such that for any $\xi_1,\xi_2\in[-a,a]$ we have :

$|h'_\pm(\xi_1)-h'_\pm(\xi_2)|\leq A|\xi_1-\xi_2|^\alpha$



For the thin profile the force acting on element dx_1 , normal to the unperturbed flow velocity direction, which is the x_1 axis, is given by

$-[\![p]\!]dx_1$ where for a function F we define $[\![F]\!]=F(x_1+0)-F(x_1-0)$ and therefore the resultant normal force is

$R_N=-\int_{-a}^{a}[\![p]\!]dx_1$ and the resultant momentum of forces with respect to centre of wing is

$$M_C=-\int_{-a}^{a}[\![p]\!]x_1 dx_1$$

If the profile is a thin plane plate we have $h'_+=h'_-=-\tan(\varepsilon)$ with ε the angle of attack and the forces are normal to the plate so the parallel to the unperturbed flow velocity direction component of resultant force is $R_P=R_N\tan(\varepsilon)$

In the bidimensional subsonic case we change the spatial variables to

$X=x_1$ and $Y=\beta x_2$ where $\beta=\sqrt{1-M_0^2}$ and we have for the velocity perturbation field

$v_1=\frac{\partial\phi}{\partial X}$, $v_2=\beta\frac{\partial\phi}{\partial Y}$ with $\frac{\partial^2\phi}{\partial X^2}+\frac{\partial^2\phi}{\partial Y^2}=0$ on $\mathbb{R}^2\setminus([-a,a]\times\{0\})$, $\phi(X,Y)=\phi(x_1,x_2)$

Also we have $\rho_0 V_0\frac{\partial\phi}{\partial X}=-p$ and at first order approximation the boundary condition is

$V_0 h'_\pm(X)=\beta\frac{\partial\phi}{\partial Y}(X,\pm 0)$ for $-a<X<a$ (11)

Consider $f=f(X)=[\![\frac{\partial\phi}{\partial X}]\!]$. Obviously $f(X)=0$ for $|X|>a$ because ϕ is harmonic on $\mathbb{R}^2\setminus([-a,a]\times\{0\})$

Let $Z=X+iY$ the complex variable and the holomorphic function on $\mathbb{C}\setminus[-a,a]$

$\Phi(Z)=\frac{\partial\phi}{\partial X}-i\frac{\partial\phi}{\partial Y}$

We have

$$[\![\Phi(Z)]\!]=\begin{cases} 0 & \text{for } |X|>a \\ f(X)-i\dfrac{V_0}{\beta}[\![h']\!] & \text{for } |X|<a \end{cases}$$

where $[\![\Phi(Z)]\!]=\Phi(X,+0)-\Phi(X,-0)$ and $[\![h']\!]=h_+(X)-h_-(X)$

According to [2] the solution for Φ is

$$\Phi(Z)=\frac{1}{2\pi i}\int_{-a}^{a}(f(t)-i\frac{V_0}{\beta}[\![h']\!])\frac{dt}{t-Z}$$ and we have the Plemelj formula :

$$\left(\frac{\partial\phi}{\partial X}-i\frac{\partial\phi}{\partial Y}\right)_{Y=\pm 0}=\pm\frac{1}{2}(f(X)-i\frac{V_0}{\beta}[\![h']\!])+\frac{1}{2\pi i}{}'\int_{-a}^{a}(f(t)-i\frac{V_0}{\beta}[\![h']\!](t))\frac{dt}{t-X} \quad (12)$$

for $|X|<a$,

where for the Hölderian function *F* we used the notation for the principal value of the improper integral :

$${}'\int_{-a}^{a}F(t)\frac{1}{t-X}dt=\lim_{\epsilon\downarrow 0}\left(\int_{-a}^{X-\epsilon}F(t)\frac{1}{t-X}dt+\int_{X+\epsilon}^{a}F(t)\frac{1}{t-X}dt\right)$$

for $|X|<a$

Separating the imaginary part in (12) and considering the boundary condition (11) we will have :

$$\frac{\beta}{\pi}{}'\int_{-a}^{a}\frac{f(t)}{t-X}dt=V_0(h'_+(X)+h'_-(X)) \text{ for } |X|<a$$

The solution for the unknown function f on $(-a,a)$ is given by

$$\beta f(X)=-\frac{v_0}{\pi}\sqrt{\frac{a-X}{a+X}}{}'\int_{-a}^{a}\sqrt{\frac{a+t}{a-t}}\frac{h'_+(t)+h'_-(t)}{t-X}dt$$ and we have

$[\![p]\!]=-\rho_0 V_0 f(X)$ for $|X|<a$

It follows now that for a thin plane plate with angle of attack ε , which gives $h'_+=h'_-=-\tan(\varepsilon)$ we have, after some integration work, the resultant of aerodynamic forces in the case of subsonic flow : $R=\dfrac{\rho_0\pi\tan(\varepsilon)V_0^2}{\sqrt{1-M_0^2}}A\bar{n}$

where A is the area of the plate and $\bar{n}=(\sin(\varepsilon),\cos(\varepsilon),0)$ is the normal on the plane rectangular , with one side parallel to j axis, plate.

If the flow is supersonic $(M_0^2<1)$, the equation for ϕ function is hyperbolic and taking (in the bidimensional case)

$\beta_1=\sqrt{M_0^2-1}$, $x=x_1$, $y=x_2$ we have the solutions for $\boldsymbol{v}=(v_1,v_2)$:

$\boldsymbol{v}_\pm=(F_\pm(x\mp\beta_1 y),\mp F_\pm(x\mp\beta_1 y))$, $p_\pm=-\rho_0 V_0 F_\pm(x\mp\beta_1 y)$

where index plus is for the solution in the upper half plane and index minus is for the solution in the lower half plane.

Indeed, the general solution for ϕ of the equation $-\beta_1^2\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0$ is

$\phi=A(x+\beta_1 y)+B(x-\beta_1 y)$ and $\boldsymbol{v}=(\dfrac{\partial\phi}{\partial x},\dfrac{\partial\phi}{\partial y})$

For any $x_0\in\mathbb{R}, y>0$ we have :

$v_1(x_0-\beta_1 y\ ,\ y)=A'(x_0)+B'(x_0-2\beta_1 y)$ and
$v_2(x_0-\beta_1 y\ ,\ y)=\beta_1 A'(x_0)-\beta_1 B'(x_0-2\beta_1 y)$

We must have $\lim_{x\to-\infty}(\beta_1 v_1+v_2)=0$ and so it follows $A'(x_0)=0$

Therefore , for $y>0$ we must take
$\boldsymbol{v}(x\ ,\ y)=(B'(x-\beta_1 y)\ ,\ -\beta_1 B'(x-\beta_1 y))$ and in the same way, we must take for $y<0$ the form of solution
$\boldsymbol{v}(x\ ,\ y)=(A'(x+\beta_1 y)\ ,\ \beta_1 A'(x+\beta_1 y))$ and the whole solution follows.
As in the subsonic case we have boundary conditions at first order of approximation which lead to

$F_{\pm}(x)=\mp\frac{V_0}{\beta_1}h'_{\pm}(x)$ for $-a<x<a$

For $|x|>a$ we take $F_{\pm}(x)=0$ because the thin profile is the only source of perturbations.
Hence the resultant normal on the unperturbed flow direction component of aerodynamic force is

$$R_N=-\frac{\rho_0 V_0^2}{\beta_1}\int_{-a}^{a}(h'_{+}+h'_{-})dx$$

For the thin plane plate with area *A* and angle of attack ε we have a resultant force normal to the plate given by the relation

$$R=\frac{2\tan(\varepsilon)\rho_0 V_0^2 A}{\sqrt{M_0^2-1}}\bar{n}\quad \text{with}\quad \bar{n}=(\sin(\varepsilon)\ ,\ \cos(\varepsilon)\ ,\ 0)$$

Regarding now the boomerang wing, considering $\boldsymbol{u_0}=\boldsymbol{w_0}+\boldsymbol{v_0}$ the unperturbed flow in the instantaneous body tied reference frame we assume as mentioned, because of the throwing conditions and aerodynamic shape of the wing that only $\boldsymbol{u_1}=\boldsymbol{u_0}-(\boldsymbol{u_0}\cdot\bar{k})\bar{k}$ is relevant for aerodynamic resultant forces and so we can consider that the aerodynamic forces system is equivalent approximatively to a resultant force ***R*** with application point at the centre of the wing which is at r_0 distance from the mass centre *C* and we have for the wing *q* (*q* = 0,1,2) :

$$\boldsymbol{R}=R_P\frac{\boldsymbol{u_1}}{\|\boldsymbol{u_1}\|}+R_N\bar{k}$$

$$R_P=c_P\frac{\boldsymbol{u_1^2}}{\sqrt{1-\frac{\boldsymbol{u_1^2}}{c^2}}}\rho_0 A$$

$$R_N=c_N\frac{\boldsymbol{u_1^2}}{\sqrt{1-\frac{\boldsymbol{u_1^2}}{c^2}}}\rho_0 A$$

$$c_P=c_N\tan(\varepsilon_0)$$

where ε_0 is the angle between the normal versor of the wings plane $\bar{k}$ and the resultant force ***R***, c_P ,c_N are aerodynamic coefficients and *A* is the reference area of the wing.
The aerodynamic shape of the wing leads us to consider that c_P, c_N = 0 if the flow comes from behind the wing $(\text{ if } \boldsymbol{u_0}\cdot T^q\bar{i}<0)$
The aerodynamic coefficients may depend on other positional variables such as the angle of attack. However , because we assumed that the *k* component of the unperturbed flow velocity is not significant we consider that the variations of these coefficients are not qualitative important .
To run the Python program for computing the trajectory of the boomerang and after that presenting

a Matplotlib animation of the boomerang flight I have taken: the angle ep= ε_0 = 0.17 rad , the radius r_0=0.2 m ; the wing referential dimensions *a*=*b*= 0.06 m ; the aerodynamic coefficient cf = 0.2 ;the ratio rt = 1 between the horizontal, in j_1 direction initial velocity of the throw component et = -30 m/s and the velocity of the wing centre by initial rotation versus the axis of direction *k* through the centre *C ;* the initial velocity vertical component zet = 2.5 m/s; the initial nutation angle tet = 1.55 rad (approximative 90°) ;the initial proper rotation angle phi = -0.034 rad. We have also considered an initial precession angle of 90°. That means we throw the boomerang as we mentioned in the beginning, having no wind velocity ws. With these parameters, the program computes an almost returning curved trajectory as we can see in the figures below.
The lower boundary of absolute sinus value of the nutation angle at which we consider a proper procedure for calculating the angular Euler velocities from the angular velocity vector components is ap = 0.01 . (If the nutation angle has at one step of integration an absolute sinus value below ap , we consider that $\dot{\phi}$ not changes value at that step and we calculate $\dot{\psi}$ from the third equation of angular velocity vector)
The program takes the proper rotation angle as integration variable (instead of time) and stops if the computed time derivative of this angle changes sign. The step of integration is the "angular spacing" parameter sp , which I have taken 40.
For preventing uncontrolled increasing of computed angular momentum during the step by step integration the program corrects the modulus of angular momentum by a formula for integration of the angular momentum modulus.
The program runs also a reduced loop of iterations in integration which contains the first four points of the trajectory at which the x coordinate becomes zero or the trajectory becomes parallel to xOz plane in the fix reference frame.

REFERENCES

[1] M. Rădoi, E. Deciu MECANICA, Ediția a-III-a revizuită, Editura Didactică și Pedagogică București
[2] Lazăr Dragoș PRINCIPIILE MECANICII MEDIILOR CONTINUE, Editura Tehnică București

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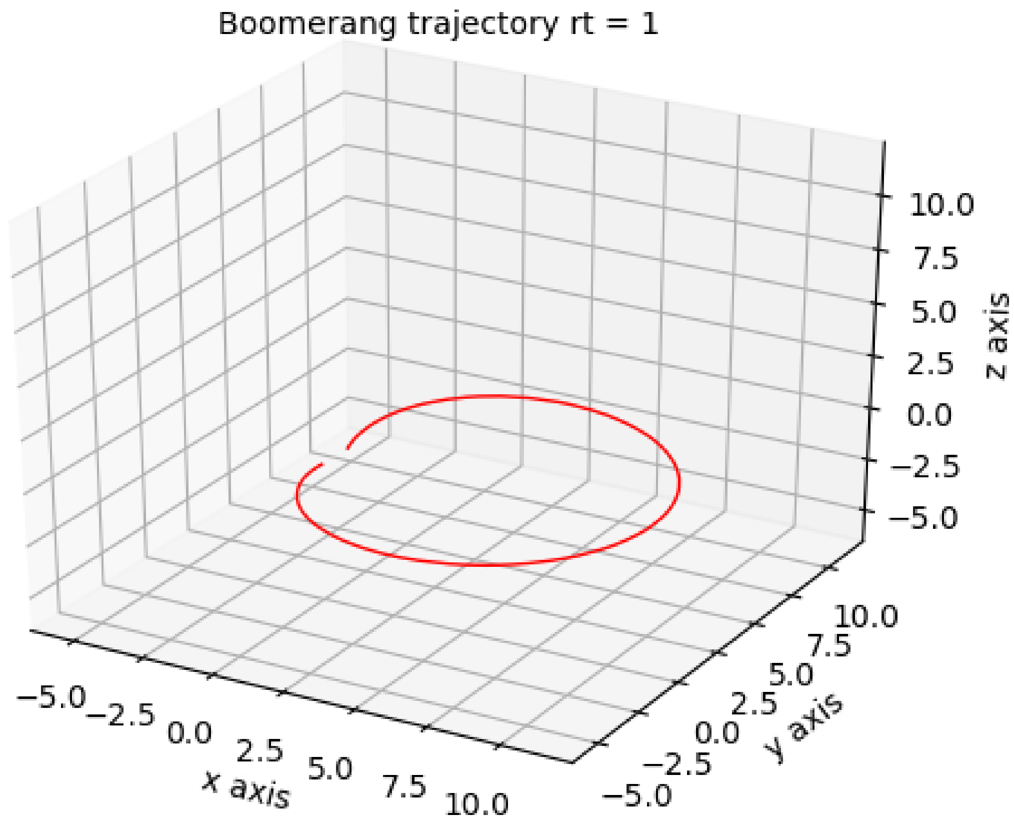
The figures at the end of the file show the computed reduced loop trajectory of the boomerang for ratios rt = 1 and respective rt = 1.1


Boomerang trajectory rt = 1
x axis
y axis
z axis
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

Boomerang trajectory rt = 1.1