Representations of the rotation group and of the restricted Lorentz group
Spin representations

For a finite dimensional vector space V, we have the general linear group
$GL(V)=\{T:V\rightarrow V|T \text{ linear homeomorphism}\}$
with the usual composition operation and topology and for a given Lie group G (as a real manifold with continuous differentiable group inversion and multiplication) we consider group representations U such that for any map of G, $h:D\rightarrow G$, U is considered to be definite on the map domain D by a function $U:D\rightarrow GL(V)$ and there exists a map
$h_0:D_0\rightarrow G$ with $\mathbb{I}_G\in h_0(D_0)$ such that for any map $h:D\rightarrow G$ of the manifold G ,
for any $R\in h(D)$ exist neighbourhoods of $\mathbb{I}_G$ and R , W_0 respective W_1 such that

$U(h^{-1}(R_0R_1))=U(h_0^{-1}(R_0))U(h^{-1}(R_1))$ for any $R_0\in W_0$, $R_1\in W_1$.
if there is no confusion we will denote $U\circ h_0^{-1}$ by U and $U\circ h^{-1}$ by U_h
Moreover we consider that U is continuous differentiable on map domain for any map of G.
In the following we will denote indexing from 1 to 3 by Latin characters and indexing from 1 to 4 by Greek characters and also use the Einstein summation convention for repeating indexes.
Let $G=\{R\in M_{3\times 3}(\mathbb{R})|R^T R=\mathbb{I}\text{ , } \det R=1 \text{ , } R=(R_{ij})\}=SO(3)$ the rotation group.
Any $R\in SO(3)$ can be written as $R=R(\varphi,n)$ a rotation around an axis of versor
$n=(n_i)$ by an angle of φ radians and we will have :
$R_{ij}=-\epsilon_{ijk}n_k\sin(\varphi)+(\delta_{ij}-n_in_j)\cos(\varphi)+n_in_j$
Obviously we have :
$R(\varphi+\delta\varphi,n)=R(\delta\varphi,n)R(\varphi,n)$
$\frac{dR}{d\varphi}(\varphi,n)=\frac{dR}{d\varphi}(0,n)R(\varphi,n)$
$R(\delta\varphi,n)=\mathbb{I}-i\,\delta\varphi n_k\bar{J}_k+O(\delta\varphi^2)$ with $(\bar{J}_k)_{ij}=-i\,\epsilon_{ijk}$ (1)
$\frac{dR}{d\varphi}(0,n)=-i\,n_k\bar{J}_k$
$R(\varphi,n)=\exp(-i\,\varphi n_k\bar{J}_k)$ (1’)
Therefore SO(3) is a 3-dimensional manifold with maps given from the parametrisation in
$(\varphi_1,\varphi_2,\varphi_3)=(\varphi n_1,\varphi n_2,\varphi n_3)$ as local coordinates and further we will take as h_0 the map from the (0, 0, 0) containing domain.
It is easy to verify that we have the commutation relations:
$[\bar{J}_i,\bar{J}_j]=i\,\epsilon_{ijk}\bar{J}_k$ where $[A,B]=AB-BA$ denotes the commutator of A and B.
Let U be a representation of SO(3) over a finite dimensional complex vector space V such that U takes unitary operators as values. We have :
$U_h(R(\varphi+\delta\varphi,n))=U(R(\delta\varphi,n))U_h(R(\varphi,n))$ if $\delta\varphi$ is small enough
and so , differentiating with respect to $\delta\varphi$ we obtain
$\frac{dU}{d\varphi}(R(\varphi,n))=\frac{dU}{d\varphi}(R(0,n))U(R(\varphi,n))$ for $R(\varphi,n)\in h_0(D_0)$

and if we define the operators J_k by $\frac{dU}{d\varphi}(R(0,n))=-i\,n_kJ_k$ we will have :

$U(R(\varphi,n))=\exp(-i\,\varphi n_kJ_k)$ for $R(\varphi,n)\in h_0(D_0)$ (2)
and
$U(R(\delta\varphi,n))=\mathbb{I}-i\,\delta\varphi n_kJ_k+O(\delta\varphi^2)$ (2’)

The representation being unitary it follows that the operators J_k must be self-adjoint.
For any $R \in SO(3)$, because $\det R = 1$ we have
$R_{jp} \epsilon_{ijk} R_{kq} = R_{im} \epsilon_{mpq}$, $R^T \bar{J}_i R = R_{im} \bar{J}_m$ and thus
$R^T \exp(-i\theta \bar{J}_j) R = \exp(-i\theta R_{jk} \bar{J}_k)$ (3) and for φ, θ small enough with $R = R(\varphi, n)$
we will have :
$U(R)^{-1} U(\exp(-i\theta \bar{J}_j)) U(R) = U(\exp(-i\theta R_{jk} \bar{J}_k))$ (4) and from (1') and (2) we obtain now
$U(R)^{-1} \exp(-i\theta J_l) U(R) = \exp(-i\theta R_{lj} J_j)$ (5)
Differentiating with respect to θ we obtain
$U(R)^{-1} J_l U(R) = R_{lj} J_j$ (5')
Taking $\varphi = \delta\varphi$ from (1) and (2') follows
$(\mathbb{I} + i\delta\varphi n_k J_k) J_l (\mathbb{I} - i\delta\varphi n_k J_k) = (\delta_{lj} - i\delta\varphi n_k (\bar{J}_k)_{lj}) J_j + O(\delta\varphi^2)$
and so, because $(\bar{J}_k)_{lj} = -i\epsilon_{klj}$ we have the commutation relations :

$[J_k, J_l] = i\epsilon_{klj} J_j$ (6)

We say that the representation U is irreducible if and only if there are no proper invariant subspaces of V ,i.e. if
V_1 is a subspace of V satisfying $U(R)(V_1) \subset V_1$ for any $R \in h_0(D_0)$ then $V_1 = \{0\}$ or $V_1 = V$
Consider now U a finite dimensional complex unitary representation of SO(3).
Because of the commutation relations (6) we find that $J^2 = J_k J_k$ commutes with all of the generators J_l and by (2) with $U(R)$ for any $R = R(\varphi, n) \in h_0(D_0)$
U being unitary J^2 is selfadjoint positive semi-definite and so it has an eigenvalue $\lambda \in \mathbb{R}_+$
For $R \in h_0(D_0)$,if $J^2 v = \lambda v$ we have $J^2 U(R) v = U(R) J^2 v = \lambda U(R) v$ and $U(R)$ leaves the eigenspace of λ invariant . Therefore , because the representation is irreducible, the eigenspace must be the whole space V.
Let denote $(J_k) = (J_x, J_y, J_z)$. We can take $j \geq 0$ such that $\lambda = j(j+1)$
J_z being self-adjoint and V finite dimensional, there will be a finite number of distinct eigenvalues of $J_z : \lambda_1 < \lambda_2 < \ldots < \lambda_p$
Let $J_+ = J_x + iJ_y$. Then if $J_z v = \mu v$ with $v \neq 0$ from (6) follows $J_z J_+ v = (\mu + 1) v$
Hence, because V is finite dimensional we can take $m_0 = \max\{m \in \mathbb{N} | J_+^m v \neq 0\}$.
Let $v_0 = J_+^{m_0} v$ and we will have $J_z v_0 = (\mu + m_0) v_0$
For $J_- = J_x - iJ_y$. Then if $J_z w = \rho w$ with $w \neq 0$ follows $J_z J_- w = (\rho - 1) J_- w$ and we take $m_1 = \max\{m \in \mathbb{N} | J_-^m v_0 \neq 0\}$, $v_k = J_-^k v_0$ for $k = \overline{0, m_1}$.
From (6) follows $J_+ J_- = J^2 - J_z^2 + J_z$ and therefore for $k = \overline{1, m_1}$ we have
$J_+ v_k = (j(j+1) - (\mu + m_0 - k + 1)^2 + (\mu + m_0 - k + 1)) v_{k-1}$ and also $J_+ v_0 = 0$
Hence the subspace generated by $v_0, v_1 \ldots, v_{m_1}$, $S = Sp[v_0, v_1 \ldots, v_{m_1}]$ is invariant under J_+, J_-, J_z and so under $U(R)$ for any $R \in h_0(D_0)$ which leads to $S = V$ and
$\{\lambda_1, \lambda_2, \ldots, \lambda_p\} = \{\mu + m_0 - m_1, \mu + m_0 - m_1 + 1, \ldots, \mu + m_0\}$, $m_1 + 1 = p$ the eigenspace for each eigenvalue $\lambda_k = \mu + m_0 - m_1 + k - 1$ being unidimensional and so we have $\alpha_k \in \mathbb{C}$ such that
$J_+ v_k = \alpha_k v_{k-1}$ for $k = \overline{1, m_1}$ and also we have $J_- v_k = v_{k+1}$ for $k = \overline{0, m_1 - 1}$,
$J_+ v_0 = 0$, $J_- v_{m_1} = 0$ and $J_+^+ = J_-$ because J_x and J_y are self-adjoint. Therefore we have
$|\alpha_k^2| \langle v_{k-1} | v_{k-1} \rangle = \langle v_k | J_- J_+ | v_k \rangle = \langle v_k | v_k \rangle (j(j+1) - (\mu + m_0 - k)(\mu + m_0 - k + 1))$ and
$\langle v_{k+1} | v_{k+1} \rangle = \langle v_k | J_+ J_- | v_k \rangle = \langle v_k | v_k \rangle (j(j+1) - (\mu + m_0 - k)(\mu + m_0 - k - 1))$
for $k = \overline{1, m_1}$ and respective $k = \overline{0, m_1 - 1}$ and
$\mu + m_0 = j$, $-\mu - m_0 + m_1 = j$
Hence we have

$m_1 = 2j$ and $-j \le \mu + m_0 - k \le j$ for $k = \overline{0, m_1}$, $dim V = 2j+1$
Unitary complex finite dimensional irreducible representations of SO(3) have 2 j + 1 dimensional
J_z having eigenvalues with one-dimensional eigenspaces : $-j, -j+1, \ldots, j-1, j$
J^2 has only eigenvalue $j(j+1)$ and j is a non-negative half-integer multiple.
If we take for V the wave functions Hilbert space of a quantum particle, because of the commutation relations for coordinates operators and momentum operators ,
$[\hat{x}_l, \hat{p}_j] = i\,\delta_{lj}\hbar$

it follows that the angular momentum operators $J_i = \frac{1}{\hbar}\hat{L}_i$ with $\hat{L} = \hat{X} \times \hat{P}$
satisfy the commutation relations (6) and therefore they can generate an unitary complex representation of SO(3). In polar coordinates (r, θ, φ) we have:

$$\frac{1}{\hbar^2}\hat{L}^2 = -\frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\varphi^2} - \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right)$$

the spherical functions operator , which has the eigenvalues $l(l+1)$ with eigenstates the spherical harmonics
$Y_l^k(\theta, \varphi) = P_l^{|k|}(\cos(\theta))\exp(i k \varphi)$ with $k, l \in \mathbb{N} |k| \le l$ and $P_l^{|k|}$ the associated Legendre polynomials. Also we will have :

$$\frac{1}{\hbar}\hat{L_+} = \exp(i\varphi)\frac{\partial}{\partial\theta} + i\cot(\theta)\exp(i\varphi)\frac{\partial}{\partial\varphi}$$

$$\frac{1}{\hbar}\hat{L_-} = -\exp(-i\varphi)\frac{\partial}{\partial\theta} + i\cot(\theta)\exp(-i\varphi)\frac{\partial}{\partial\varphi}$$

$$\frac{1}{\hbar}\hat{L_z} = -i\frac{\partial}{\partial\varphi}$$

where we have taken
$z = r\cos(\theta)$, $y = r\sin(\theta)\sin(\varphi)$, $x = r\sin(\theta)\cos(\varphi)$
The eigenstates of the $l(l+1)$ generate (for constant r) the invariant subspace of the irreducible spin l representation.

Let (σ_k) be the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For $M \in SU(2) = \{S \in M_{2\times 2}(\mathbb{C}) | S^+ S = \mathbf{I} , det S = 1\}$ we have uniquely determined $(\alpha_k) \in \mathbb{C}^3$ and $\alpha_0 \in \mathbb{C}$ such that $M = \alpha_0 \mathbf{I} - i\,\alpha_k \sigma_k$, because $(\mathbf{I}, \sigma_1, \sigma_2, \sigma_3)$ provide a basis for the complex vector space $M_{2\times 2}(\mathbb{C})$

For $a = \Re(\alpha_0)$, $b = \Im(\alpha_0)$, $\vec{X} = \Re(\vec{\alpha})$, $\vec{Y} = \Im(\vec{\alpha})$ the conditions $M \in SU(2)$ lead to $a^2 + b^2 + \vec{X}^2 + \vec{Y}^2 = 1$ and $a^2 + \vec{X}^2 - b^2 - \vec{Y}^2 = 1$ and so we have a versor (n_k) and an angle $\frac{\theta}{2}$ uniquely determining $a = \cos(\frac{\theta}{2})$, $\vec{X} = n\sin(\frac{\theta}{2})$, $b = 0$, $\vec{Y} = 0$

Therefore SU(2) is a 3-dimensional Lie group with local mappings given by the parametrisation
$(\varphi n_k) \in \mathbb{R}^3$, $h((\varphi n_k)) = \exp(-i\frac{1}{2}n_k\sigma_k) = \cos(\frac{\varphi}{2})\mathbf{I} - i\sin(\frac{\varphi}{2})n_k\sigma_k$
We can verify that we have a local diffeomorphism
$T : SU(2) \rightarrow SO(3)$ which in any map parametrisation (φn_k) has the expression
$T(\exp(-i\frac{1}{2}\varphi n_k \sigma_k)) = R(\varphi, n)$
Moreover, considering the factor group
$SU(2)/\{-\mathbf{I}, \mathbf{I}\}$ with the projection $p : SU(2) \rightarrow SU(2)/\{-\mathbf{I}, \mathbf{I}\}$ we have that

$p \circ T^{-1}$ is well defined as diffeomorphism from $SO(3)$ to $SU(2)/\{-\mathbf{I},\mathbf{I}\}$ which has a differential manifold structure that can be considered as induced by the local diffeomorphism T.
$SU(2)$ is a double covering of $SO(3)$, for any $R(\varphi,n)$ corresponding
$\pm(\cos(\frac{\varphi}{2})\mathbf{I}-i\sin(\frac{\varphi}{2})n_k\sigma_k)$ because we have $R(\varphi,n)=R(\varphi+2\pi,n)$
For $R=R(\varphi,n)\in SO(3)$, $S\in SU(2)$, $T(S)=R$ we have that
$S^{-1}\sigma_k S=R_{kj}\sigma_j$ and so if $S_i\in SU(2)$ satisfies $T(S_i)=R_i\in SO(3)$ for $i=1,2$ then
for $S=S_1S_2$, $R=R_1R_2$ we have that $S^{-1}\sigma_k S=R_{kj}\sigma_j$ with $k=1,2,3$
If $W\in SU(2)$ satisfies $T(W)=R$ we will have also $W^{-1}\sigma_k W=R_{kj}\sigma_j$ and therefore
for $H=SW^{-1}$ we have $H\sigma_k=\sigma_k H$ with $k=1,2,3$
Thus $(\mathbf{I},\sigma_1,\sigma_2,\sigma_3)$ being a basis of $M_{2\times 2}(\mathbb{C}),H$ commutes with any 2x2 complex matrix so exists $\lambda\in\mathbb{C}$ such that $H=\lambda\mathbf{I}$ and because $\det H=1$ follows $H=\pm\mathbf{I}$
therefore $S=\pm W$ and because $T(W)=T(-W)$ we conclude that
$T(S_1)T(S_2)=T(S_1S_2)$ for any $S_1,S_2\in SU(2)$ and $p\circ T^{-1}$ is a groups isomorphism.

Let U the so called spin $\frac{1}{2}$ representation $U(R(\varphi,n))=\exp(-i\frac{1}{2}\varphi n_k\sigma_k)$
For any map h of $SO(3)$ we have obviously $T(U\circ h^{-1}(R))=R$ and so
$T(U\circ h^{-1}(R_0R_1))=T(U\circ h_0^{-1}(R_0))T(U\circ h^{-1}(R_1))$ and as we have proven above it follows
$U_h(R_0R_1)=\pm U(R_0)U_h(R_1)$ for R_0,R_1 in some neighbourhoods of $\mathbf{I}$ respective R (*)
Because U_h and $U\circ h_0$ are continuous in neighbourhoods of R and respective $\mathbf{I}$ and
$U\circ h_0^{-1}(\mathbf{I})=\mathbf{I}$, from the relation (*) we can derive the condition for U to be indeed a representation of SO(3).

For U^i a $GL(V_i)$ valued representation of $SO(3)$ with $i=\overline{1,n}$ we can consider the
$GL(\bigotimes_{i=1}^{n}V_i)$ valued representation which in any map $h:D\to SO(3)$ has the expression
$U_h(R)(\varphi_1\otimes\varphi_2\ldots\otimes\varphi_n)=U_h^1(R)\varphi_1\otimes U_h^2(R)\varphi_2\otimes\ldots\otimes U_h^n(R)\varphi_n$
If we denote the generators of U^k by $J_{k,i}i=1,2,3$ then for the generators J_i of U we have
$J_i=\sum_{k=1}^{n}\mathbf{I}\otimes\ldots\otimes J_{k,i}\otimes\ldots\otimes\mathbf{I}$ and so J_z carries eigenvalues $m_1+m_2+\ldots+m_n$
with $m_k\in\{-j_k,\ldots,j_k\}$ if U^k is a spin j_k representation for $k=\overline{1,n}$
Take now $n=2j$ and $U^k=U^{(1)}$,the same spin $\frac{1}{2}$ representation , valued on $GL(V^{(1)})$
having generators $J_i^{(1)}$ with eigenstates e_+,e_- for eigenvalues $\frac{1}{2}$ respective $-\frac{1}{2}$ of $J_z^{(1)}$
We can consider the subspace of symmetric tensors of the tensorial product space
$V^{(n)}=\bigotimes_{k=1}^{2j}V^{(1)}$ namely
$S=\{\sum_{i1,i2,\ldots,in=\pm}a_{i1i2\ldots in}\sum_{\tau\in S_n}e_{i\tau(1)}\otimes e_{i\tau(2)}\otimes\ldots\otimes e_{i\tau(n)}|a_{i1i2\ldots in}\in\mathbb{C}\text{ for }i1,i2,\ldots,in=\pm\}$
The product representation is $U^{(n)}$ with generators $J_i^{(n)}$
The subspace S is invariant under $U^{(n)}$ carries the eigenstate $e_+\otimes e_+\otimes\ldots\otimes e_+$
of eigenvalue j of $J_z^{(n)}$ and has dimension $n+1=2j+1$ and therefore the restriction of
$U^{(n)}$ to S must be a spin j irreducible representation of $SO(3)$.
In the same way we conclude that the representation given by

$U(R)=R$ for any $R\in SO(3)$ is a spin 1 irreducible representation
and the representation given by
$U(R)((\varepsilon_{ij})_{i,j=1,2,3})=(R_{ki}R_{lj}\varepsilon_{ij})_{k,l}$ with invariant space
$V=\{\varepsilon\in M_{3\times3}(\mathbb{C})|\varepsilon_{ij}=\varepsilon_{ji}\ ,\ \varepsilon_{kk}=0 \text{ with } i,j=1,2,3\}$, the symmetric traceless tensors, is a spin 2 representation.

Consider now $G=SO^{+}(3,1)$ the restrict Lorentz group (by suitable measuring units for time we can consider the speed of light constant to be c = 1) and the Minkowski space with pseudo-metric $(\eta^{\alpha\beta})$, $\eta^{ij}=-\delta_{ij}$, $\eta^{i4}=\eta^{4i}=0$, $\eta^{44}=1$

For any $M\in G$ we have uniquely determinated $B=B(\chi,q)$, $R=R(\theta,n)$ with $n=(n_i)$, $q=(q_i)$ versors and $\chi,\theta\in\mathbb{R}$ such that $M=BR$

$R_{ij}=-\epsilon_{ijk}n_k\sin(\theta)+(\delta_{ij}-n_in_j)\cos(\theta)+n_in_j$, $R_{i4}=R_{4i}=0$, $R_{44}=1$

$B_{ij}=\delta_{ij}+(\cosh(\chi)-1)q_iq_j$, $B_{i4}=B_{4i}=-q_i\sinh(\chi)$, $B_{44}=\cosh(\chi)$

(see Chap. Special relativity. Lorentz transformation)$v_i=q_i\tanh(\chi)$

$G=SO^{+}(3,1)$ is therefore a 6-dimensional Lie group with maps by parametrisation in $((\chi q_i),(\theta n_i))$ and as the map h_0 we will take the map which contains $(0)\in\mathbb{R}^6$ in its domain.

We can verify that

$B(\chi+\delta\chi,q)=B(\delta\chi,q)B(\chi,q)$ (7)

$R(\theta+\delta\theta,n)=R(\delta\theta,n)R(\theta,n)$ (8)

and we can define $(\bar{J}_i)$, $(\bar{K}_i)$ such that

$n_k\bar{J}_k=\dfrac{dR}{d\theta}(0,n)$, $q_k\bar{K}_k=-\dfrac{dB}{d\chi}(0,q)$ with

$(\bar{J}_i)_{jk}=-\epsilon_{ijk}$, $(\bar{J}_i)_{4\alpha}=(\bar{J}_i)_{\alpha 4}=0$, $(\bar{K}_i)_{jk}=0$, $(\bar{K}_i)_{4j}=(\bar{K}_i)_{j4}=\delta_{ij}$, $(\bar{K}_i)_{44}=0$

and so we will have :

$B(\chi,q)=\exp(-\chi q_k\bar{K}_k)$, $R(\theta,n)=\exp(\theta n_k\bar{J}_k)$ (9)

$M(\delta\chi,q;\delta\theta,n)=B(\delta\chi,q)R(\delta\theta,n)=\mathbf{I}-\delta\chi q_k\bar{K}_k+\delta\theta n_k\bar{J}_k+O(\varepsilon^2)$ (9')

for $\delta\chi$, $\delta\theta\in O(\varepsilon)$

$[\bar{J}_i,\bar{J}_j]=\epsilon_{ijk}\bar{J}_k$, $[\bar{K}_i,\bar{K}_j]=-\epsilon_{ijk}\bar{J}_k$, $[\bar{J}_i,\bar{K}_j]=\epsilon_{ijk}\bar{K}_k$ (10)

For a representation U of $SO^{+}(3,1)$ we can define (J_i) , (K_i) such that

$n_kJ_k=\dfrac{dU}{d\theta}(R(0,n))$, $q_kK_k=-\dfrac{dU}{d\chi}(B(0,q))$ and we will have:

$U(B(\chi,q))=\exp(-\chi q_kK_k)$, $U(R(\theta,n))=\exp(\theta n_kJ_k)$ (11)

$U(M(\delta\chi,q;\delta\theta,n))=\mathbf{I}-\delta\chi q_kK_k+\delta\theta n_kJ_k+O(\varepsilon^2)$ (11')

for $\delta\chi$, $\delta\theta\in O(\varepsilon)$

Let $A_l(\theta)=R(-\theta,n)\bar{J}_lR(\theta,n)$. Then from (9) and (10) follows

$\dfrac{dA_l}{d\theta}=\epsilon_{lkj}A_j=(\bar{J}_k)_{lj}A_j$ and because $A_l(0)=\bar{J}_l$ we have the solution

$A_l=R_{lj}\bar{J}_j$ where $R=R(\theta,n)$ and so we have

$R^{-1}\exp(\varphi\bar{J}_l)R=\exp(\varphi R_{lj}\bar{J}_j)$

Therefore, according (9) and (11) , for θ , φ small enough we obtain

$U(R)^{-1}\exp(\varphi J_l)U(R)=\exp(\varphi R_{lj}J_j)$ and taking the second order approximation in φ:

$U(R)^{-1}J_l U(R)=R_{lj}J_j$ and so for $\theta=\delta\theta$ follows
$(\mathbf{I}-\delta\theta n_k J_k)J_l(\mathbf{I}+\delta\theta n_k J_k)=(\delta_{lj}-\delta\theta n_k\epsilon_{ljk})J_j+O(\delta\theta^2)$
$[J_l,J_k]=\epsilon_{lkj}J_j$ (12)

In the same way, taking $A_l(\theta)=R(-\theta,n)\bar{K}_l R(\theta,n)$ we obtain
$R^{-1}\exp(-\chi\bar{K}_l)R=\exp(-\chi R_{lj}\bar{K}_j)$ with $R=R(\theta,n)$ and further, if $\theta=\delta\theta$ is small enough :
$(\mathbf{I}-\delta\theta n_k J_k)K_l(\mathbf{I}+\delta\theta n_k J_k)=(\delta_{lj}-\delta\theta n_k\epsilon_{klj})K_j+O(\delta\theta^2)$
(n_k) being an arbitrary versor, we will have
$[J_k,K_l]=\epsilon_{klj}K_j$ (13)

We take now
$A_l(\chi)=B(-\chi,q)\bar{K}_l B(\chi,q)$, $C_l(\chi)=B(-\chi,q)\bar{J}_l B(\chi,q)$ and we have from (9) and (10)

$$\frac{dA_l}{d\chi}=-q_k\epsilon_{klj}C_j$$

$$\frac{dC_l}{d\chi}=-q_k\epsilon_{klj}A_j$$

Therefore , for $B=B(\chi,q)$ and $R=R(\chi,q)$ the solution
$B^{-1}(\bar{K}_l+\bar{J}_l)B=R_{lj}(\bar{K}_j+\bar{J}_j)$ (14)
From (10) we obtain $[\bar{K}_i+\bar{J}_i,\bar{K}_j+\bar{J}_j]=0$, $[\bar{K}_i,\bar{J}_i]=0$ for $i,j=1,2,3$ and so we have:

$$\exp(\chi'(\bar{K}_l+\bar{J}_l))=\exp(\chi'\bar{K}_l)\exp(\chi'\bar{J}_l) \quad \text{and}$$

$$\exp(\chi' R_{lj}(\bar{K}_j+\bar{J}_j))=\prod_{j=1}^{3}\exp(\chi' R_{lj}\bar{K}_j)\exp(\chi' R_{lj}\bar{J}_j)$$

Multiplying (14) by χ' , exponentiating , applying U for small enough χ and χ' and after that considering (11) we obtain now:

$$U(B)^{-1}\exp(\chi' K_l)\exp(\chi' J_l)U(B)=\prod_{j=1}^{3}\exp(\chi' R_{lj}K_j)\exp(\chi' R_{lj}J_j)$$

Taking the second order approximation in χ' we obtain, for small enough χ that:
$U(B)^{-1}(K_l+J_l)U(B)=R_{lj}(K_j+J_j)$ and for $\chi=\delta\chi$
$(\mathbf{I}+\delta\chi q_k K_k)(K_l+J_l)(\mathbf{I}-\delta\chi q_k K_k)=(\delta_{lj}-\delta\chi q_k\epsilon_{klj})(K_j+J_j)$
With (13) we can now conclude that
$[K_k,K_l]=-\epsilon_{klj}J_j$ (15)
We have therefore the commutation relations (12) , (13) , (15) for the generators.

Consider now the Dirac equation for a four component wave function $\psi=(\psi_\alpha)$ (as a column vector) of a mass *m* particle :
$i\gamma^\mu\partial_\mu\psi-m\psi=0$
with the 4x4 matrices

$$\gamma^k=\begin{pmatrix}0 & \sigma_k\\ -\sigma_k & 0\end{pmatrix},\ \gamma^4=\begin{pmatrix}\mathbf{I} & 0\\ 0 & -\mathbf{I}\end{pmatrix}$$

Under a Lorentz transformation $M=(M_{\alpha\beta})$ with
$x'^\mu=M_{\mu\delta}x^\delta$, $(x^\mu)=(x,y,z,t)$, $(x'^\mu)=(x',y',z',t')$ (we consider the speed of light c = 1)
we suppose that the wave function transforms like

$\psi'_{\alpha}=S_{\alpha\delta}\psi_{\delta}$

We have $M_{\nu\mu}\partial_{\nu}'=\partial_{\mu}$, $\gamma^{\mu}M_{\nu\mu}\partial_{\nu}'S^{-1}\psi'=mS^{-1}\psi'$ and so requiring Lorentz invariance of the Dirac equation we come to

$S^{-1}\gamma^{\nu}S=M_{\nu\mu}\gamma^{\mu}$

We can verify that:

$\gamma^{\alpha}\gamma^{\beta}+\gamma^{\beta}\gamma^{\alpha}=2\eta^{\alpha\beta}$ (16)

Considering (16), for $M=B(\chi,q)$ we can take

$S=P(\chi,q)=\cosh(\frac{\chi}{2})\mathbf{I}+\sinh(\frac{\chi}{2})q_k\gamma^k\gamma^4$

and for $M=R(\theta,n)$ we can take

$S=Q(\theta,n)=\cos(\frac{\theta}{2})\mathbf{I}+\frac{1}{2}\sin(\frac{\theta}{2})n_k\epsilon_{kij}\gamma^j\gamma^j$

Let $SL(2,\mathbb{C})=\{S\in M_{2\times2}(\mathbb{C})|det\,S=1\}$

Since $(\mathbf{I},\sigma_1,\sigma_2,\sigma_3)$ is a basis of $M_{2\times2}(\mathbb{C})$ we have $\alpha_0,\alpha_1,\alpha_2,\alpha_3\in\mathbb{C}$, uniquely determined for $S\in SL(2,\mathbb{C})$ such that $S=\alpha_0\mathbf{I}+\alpha_k\sigma_k$ (17)

and $\alpha_0^2-\vec{\alpha}^2=1$ which leads to

$(\Re\alpha_0)^2-(\Im\alpha_0)^2=(\Re\vec{\alpha})^2-(\Im\vec{\alpha})^2+1$ (18)

$(\Re\alpha_0)(\Im\alpha_0)=(\Re\vec{\alpha})(\Im\vec{\alpha})$ (18')

If we suppose now that $S=(a\mathbf{I}-X_k\sigma_k)(b\mathbf{I}-iY_k\sigma_k)$ (19)

with $a,b\in\mathbb{R}$, $a\geq1$, $(X_k),(Y_k)\in\mathbb{R}^3$

$a^2-\vec{X}^2=1$ (19')

$b^2+\vec{Y}^2=1$ (19'') , then (17) leads to

$ab+i\vec{X}\vec{Y}=\alpha_0$ (20) and

$b\vec{X}+ia\vec{Y}+\vec{X}\times\vec{Y}=-\vec{\alpha}$ (21) , or , by taking real and imaginary parts :

$ab=\Re\alpha_0$ (22)

$b\vec{X}+\vec{X}\times\vec{Y}=\Re\vec{\alpha}$ (23)

$\vec{X}\vec{Y}=\Im\alpha_0$ (24)

$a\vec{Y}=-\Im\vec{\alpha}$ (25)

Also from (17) we have :

$b\mathbf{I}-iY_k\sigma_k=(a\mathbf{I}+X_k\sigma_k)(\alpha_0\mathbf{I}+\alpha_k\sigma_k)$ and so

$b=a\alpha_0+\vec{\alpha}\vec{X}$ (26)

$\vec{Y}=i\alpha_0\vec{X}+ia\vec{\alpha}-\vec{X}\times\vec{\alpha}$ (27)

1. If $(\Re\vec{\alpha})\times(\Im\vec{\alpha})=0$

   1.1 if $\Im\vec{\alpha}=0$ we obtain $\vec{Y}=0$ from (25) and so, from (19'') $b^2=1$

By (18) and (18') we will have in this case $\Im\alpha_0=0$ and taking the real part of (27) it follows

$\vec{X}\times\Re\vec{\alpha}=0$, $\vec{X}=\lambda\Re\vec{\alpha}$ with $\lambda\in\mathbb{R}$

From (26) we have now $b=a\alpha_0+\lambda(\Re\vec{\alpha})^2$ and multiplying by a , using (22) we have:

$\alpha_0(1-a^2)=\lambda\vec{\alpha}^2$ and so with (19') follows $-\alpha_0\lambda^2\vec{\alpha}^2=\lambda\vec{\alpha}^2a$ (28)

If in this case $\alpha_0^2=1$ from (18) we will have $\vec{\alpha}=0$ and so $\vec{X}=0$ and by (19') and (22)

$a=1$, $b=\alpha_0a$. Hence $a,b,\vec{X},\vec{Y}$ are uniquely determined from (19) by $\alpha_0,\vec{\alpha}$

If in this case $\alpha_0^2\neq1$ from (18) follows $\Re\vec{\alpha}\neq0$ and from (22) and (19') follows

$a^2\neq1$ and $\vec{X}\neq0$. Therefore $\lambda\neq0$ and (28) leads to $-\lambda\alpha_0=a$

and so, by (19') and (18) $a^2=(\Re\alpha_0)^2=1+(\Re\vec{\alpha})^2>1$ provides the correct uniquely determination of $a,b,\vec{X},\vec{Y}$ from (19) by $\alpha_0,\vec{\alpha}$.

1.2. If $\Im\vec{\alpha}\neq 0$ follows

$\Re\vec{\alpha}=\lambda\Im\vec{\alpha}$ with $\lambda\in\mathbb{R}$ and because from (25) and (27) we have

$(a^2-1)\Im\vec{\alpha}=-a\Im\alpha_0\vec{X}-a\vec{X}\times\Re\vec{\alpha}$, we will also have

$a(\vec{X}\times\Re\vec{\alpha})^2=0$ and $(a^2-1)\Im\vec{\alpha}=-a\Im\alpha_0\vec{X}$ (29) which by (19') leads to

$(a^2-1)^2(\Im\vec{\alpha})^2=a^2(a^2-1)(\Im\alpha_0)^2$ (30)

1.2.1 If $(\Re\alpha_0)(\Im\alpha_0)=0$

In this subcase, from (18') follows $\Re\vec{\alpha}=0$ and with (23) and (25) we obtain

$b\vec{X}^2=0$ and $\vec{X}=\mu\vec{Y}$, $\mu\in\mathbb{R}$

From (22), (25) , (19'') and (18) we have $a^2=(\Re\alpha_0)^2+(\Im\vec{\alpha})^2=1+(\Im\alpha_0)^2+(\Re\vec{\alpha})^2\geq 1$

From (24) and (25) we have $\mu(\Im\vec{\alpha})^2=a^2\Im\alpha_0$ and so $a,b,\vec{X},\vec{Y}$ are correctly uniquely determined.

1.2.2 If $(\Re\alpha_0)(\Im\alpha_0)\neq 0$

In this subcase, (24) leads to

$\vec{X}\neq 0$ and so, by (19') $a^2\neq 1$ and from (30) follows $a^2((\Im\vec{\alpha})^2-(\Im\alpha_0)^2)=(\Im\alpha_0)^2$ (31)

In this case $(\Re\vec{\alpha})^2(\Im\vec{\alpha})^2=((\Re\vec{\alpha})(\Im\vec{\alpha}))^2$ and therefore, by (18) and (18') taking

$\mu^2=\dfrac{(\Im\alpha_0)^2}{(\Im\vec{\alpha})^2}$ we obtain $(1-\mu^2)((\Re\vec{\alpha})^2+(\Im\alpha_0)^2+1)=1$ and so $\mu^2<1$

Hence, by (31) , (29) , (22) , and (25) a,b , $\vec{X}$, $\vec{Y}$ are again correctly uniquely determined.

2. If $(\Re\vec{\alpha})\times(\Im\vec{\alpha})\neq 0$

we have $\lambda,\mu,\rho\in\mathbb{R}$ such that

$\vec{X}=\lambda\Re\vec{\alpha}+\mu\Im\vec{\alpha}+\rho(\Re\vec{\alpha})\times(\Im\vec{\alpha})$ the relations (25) , (21) and (24) leading to

$\lambda\Re\alpha_0+\rho(\Im\vec{\alpha})^2=-a$ (32)

$\mu\Re\alpha_0-\rho(\Re\vec{\alpha})(\Im\vec{\alpha})=0$ (33)

$\lambda-\rho\Re\alpha_0=0$ (34)

$\lambda(\Re\vec{\alpha})(\Im\vec{\alpha})+\mu(\Im\vec{\alpha})^2=-\Im\alpha_0$ (35)

From (22) , (25) , (19'') and (18) we have $a^2=(\Re\alpha_0)^2+(\Im\vec{\alpha})^2=1+(\Im\alpha_0)^2+(\Re\vec{\alpha})^2\geq 1$ which determines correctly

$a\geq 1$ and now (32),(34) and (35) determine λ,μ,ρ and therefore $\vec{X}$; (25) determines $\vec{Y}$

$a,b,\vec{X},\vec{Y}$ are correctly uniquely determinated from (19) by α_0 and $\vec{\alpha}$

Taking $a=\cosh(\frac{\chi}{2})$, $\vec{X}=\sinh(\frac{\chi}{2})q$, $b=\cos(\frac{\theta}{2})$, $\vec{Y}=\sin(\frac{\theta}{2})n$ with versors q,n ,

we see that $SL(2,\mathbb{C})$ can be considered as a 6-dimensional Lie group with mappings given by local parametrisation in

$((\chi q_k),(\theta n_k))\in\mathbb{R}^6$, $h((\chi q_k),(\theta n_k))=\exp(-\frac{1}{2}\chi q_k\sigma_k)\exp(-i\frac{1}{2}\theta n_k\sigma_k)$,

because we can easily verify by differentiation and same initial conditions that

$\cosh(\frac{\chi}{2})\mathbf{I}-\sinh(\frac{\chi}{2})q_k\sigma_k=\exp(-\frac{1}{2}\chi q_k\sigma_k)$ and

$\cos(\frac{\theta}{2})\mathbf{I}-i\sin(\frac{\theta}{2})n_k\sigma_k=\exp(-i\frac{1}{2}\theta n_k\sigma_k)$

We define $T:SL(2,\mathbb{C})\rightarrow SO^+(3,1)$ and $H:SL(2,\mathbb{C})\rightarrow M_{4\times 4}(\mathbb{C})$ such that if

$S=\exp(-\frac{1}{2}\chi q_k\sigma_k)\exp(-i\frac{1}{2}\theta n_k\sigma_k)$ then

$T(S)=B(\chi,q)R(\theta,n)$ and $H(S)=P(\chi,q)Q(\theta,n)$

For S_1 , $S_2\in SL(2,\mathbb{C})$ we can verify that:

$H(S_i)^{-1}\gamma^\mu H(S_i)=(T(S_i))_{\mu\nu}\gamma^\nu$ for $i=1,2$, $\mu=\overline{1,4}$ and therefore

$(H(S_1)H(S_2))^{-1}\gamma^\mu(H(S_1)H(S_2))=(T(S_1)T(S_2))_{\mu\nu}\gamma^\nu$

Let be S such that $T(S)=B(\chi,q)R(\theta,n)=T(S_1)T(S_2)$. Then we can have only

$S=\pm\exp(-\frac{1}{2}\chi q_k\sigma_k)\exp(-i\frac{1}{2}\theta n_k\sigma_k)$ and we have also:

$H(S)^{-1}\gamma^\mu H(S)=(T(S_1)T(S_2))_{\mu\nu}\gamma^{n\gamma}$ and for $W=H(S)(H(S_1)H(S_2))^{-1}$ we will have

$\gamma^\mu W=W\gamma^\mu$ for $\mu=\overline{1,4}$ (36)

We take $W=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$ with $A,B,C,D \in M_{2\times 2}(\mathbb{C})$

Taking $\mu=4$ in (36) we obtain $B=-B$ and $C=-C$ and so $B=C=0$

For $\mu=i$ in (36) follows $A\sigma_i=\sigma_i D$ (37)

From (36) we obtain $W\gamma^j\gamma^j=\gamma^j\gamma^j W$ and so, because for $i\neq j$ we have

$\gamma^j\gamma^j=\begin{pmatrix} -i\epsilon_{ijk}\sigma_k & 0 \\ 0 & -i\epsilon_{ijk}\sigma_k\end{pmatrix}$ it follows

$A\sigma_k=\sigma_k A$ and $D\sigma_k=\sigma_k D$ (38)

Hence, $(\mathbf{I},\sigma_1,\sigma_2,\sigma_3)$ being a basis of $M_{2\times 2}(\mathbb{C})$, (37) and (38) lead to

$A=D=\lambda\mathbf{I}$ with $\lambda\in\mathbb{C}$ and so $W=\lambda\mathbf{I}$, $H(S)=\lambda H(S_1)H(S_2)$ (39)

For the subspace of $\mathbb{C}^4$ (cosidered as column vectors), namely $K=\{(X,X)\in\mathbb{C}^2\times\mathbb{C}^2|\}$ we can verify that for any $S_0\in SL(2,\mathbb{C}), Z=(X,X)\in K$ we have $H(S_0)Z=(S_0X,S_0X)$

Therefore, from (39) we obtain

$S=\lambda S_1S_2$ and because $det\,S=det\,S_1=det\,S_2=1$ it follows $\lambda=\pm 1$

Obviously $T(S)=T(-S)$ for any $S\in SL(2,\mathbb{C})$ and so $T(S_1S_2)=T(S_1)T(S_2)$

Thus we have a well defined groups isomorphism

$p\circ T^{-1}:SO^+(3,1)\rightarrow SL(2,\mathbb{C})/\{-\mathbf{I},\mathbf{I}\}$ where *p* is the projection operator

$p:SL(2,\mathbb{C})\rightarrow SL(2,\mathbb{C})/\{-\mathbf{I},\mathbf{I}\}$

Moreover, *T* is a local diffeomorphism , is a double covering of $SO^+(3,1)$ by $SL(2,\mathbb{C})$ and determines also the differential structure of $SL(2,\mathbb{C})/\{-\mathbf{I},\mathbf{I}\}$

Considering $F=(p\circ T^{-1})^{-1}$ the inverse group isomorphism defined above we have that U is a representation of $SO^+(3,1)$ if and only if $U\circ F$ is a representation of $SL(2,\mathbb{C})/\{-\mathbf{I},\mathbf{I}\}$.

By composition with the projection operator at left, any representation of $SL(2,\mathbb{C})/\{-\mathbf{I},\mathbf{I}\}$ determines a representation of $SL(2,\mathbb{C})$

Consider now the functions $U:D\rightarrow SL(2,\mathbb{C})$ defined for any map $h:D\rightarrow SL(2,\mathbb{C})/\{-\mathbf{I},\mathbf{I}\}$ such that $U((\chi q_k),(\theta n_k))=\exp(-\frac{1}{2}\chi q_k\sigma_k)\exp(-i\frac{1}{2}\theta n_k\sigma_k)$ for $((\chi q_k),(\theta n_k))\in D$

We have that $T(U\circ h^{-1}(\hat{S}))=R$ for any $R\in SO^+(3,1)$ where $\hat{S}=p\circ T^{-1}(R)$

Therefore $T(U_h(\hat{S}_0\hat{S}_1))=T(U(\hat{S}_0))T(U_h(S_1))$ and so, as already proven above, we must have $U_h(\hat{S}_0\hat{S}_1)=\pm U(\hat{S}_0)U_h(\hat{S}_1)$ for $\hat{S}_0$, $\hat{S}_1$ in some neighbourhoods of $\mathbf{I}$ respective $\hat{S}\in h(D)$

Because $U\circ h^{-1}$ and $U\circ h_0^{-1}$ are continuous, if these neighbourhoods, W_0 respective W_1 , are connected then $U_h(\hat{S}_0\hat{S}_1)=U(\hat{S}_0)U_h(\hat{S}_1)$ for $(\hat{S}_0,\hat{S}_1)\in W_0\times W_1$

Hence if $\bar{U}$ is a representation of $SL(2,\mathbb{C})$ then $\bar{U}\circ U$ is a representation of $SL(2,\mathbb{C})/\{-\mathbf{I},\mathbf{I}\}$.

Therefore any representation of $SL(2,\mathbb{C})$ determines a representation of $SL(2,\mathbb{C})/\{-\mathbf{I},\mathbf{I}\}$ and backwards.

Determining irreducible representations of $SO^{+}(3,1)$ reduces to determining irreducible representations of $SL(2,\mathbb{C})$.

Let U be a representation of $SL(2,\mathbb{C})$. We denote

$A(\chi,q)=\exp(-\frac{1}{2}\chi q_k\sigma_k)$; $C(\theta,n)=\exp(-i\frac{1}{2}\theta n_k\sigma_k)$ and we will have

$A(\chi+\delta\chi,q)=A(\delta\chi,q)A(\chi,q)$; $C(\theta+\delta\theta,n)=C(\delta\theta,n)C(\theta,n)$ (40)

As we mentioned , we denote by U the same function $U\circ h_0^{-1}$ where $h_0:D_0\rightarrow GL(V)$ is the map around the origin from the representation definition.

In the same way as we proven in the case of $SO^{+}(3,1)$, considering the relations (40), if we define $(M_k),(N_k)$ by

$\frac{dU}{d\chi}(A(0,q_k))=-q_kM_k$, $\frac{dU}{d\theta}(C(0,n))=-in_kN_k$

$U(A(\chi,q))=\exp(-\chi q_kM_k)$, $U(C(\theta,n))=\exp(-i\theta n_kN_k)$ –

We will in addition suppose that the functions defined in $\chi+i\theta\in\mathbb{C}$ by

$f_j(\chi+i\theta)=U(A(\chi,(\delta_{jk}))C(\theta,(\delta_{jk})))=U(\exp(-\frac{1}{2}(\chi+i\theta)\sigma_j))$ are complex differentiable, or that the function defined on the complex variables (α_k)

$F((\alpha_k))=U(\sqrt{1+\vec{\alpha}^2}\,\mathbb{I}+\alpha_k\sigma_k)$ is complex differentiable in each variable α_k in some neighbourhood of (0 , 0, 0).

We can prove that we have $f_j(\chi+i\theta)=U(\cosh(\frac{1}{2}(\chi+i\theta))\mathbb{I}-\sinh(\frac{1}{2}(\chi+i\theta))\sigma_j)$ and so any of these two suppositions will lead to $M_k=N_k$.

Let $E_l(\theta)=\frac{1}{2}C(-\theta,n)\sigma_lC(\theta,n)$ and considering the commutation relations satisfied by $(\frac{1}{2}\sigma_k)$ we obtain $\frac{dE_k}{d\theta}=-n_k\epsilon_{klj}E_j$ and so we have the solution

$E_l=R_{lj}\frac{1}{2}\sigma_j$. Therefore for $\delta\chi,\delta\theta$ small enough we will have:

$C(-\delta\theta,n)\exp(-\frac{1}{2}\delta\chi\sigma_l)C(\delta\theta,n)=\exp(-\frac{1}{2}\delta\chi R_{lj}\sigma_j)$ and

$U(C)^{-1}\exp(-\delta\chi M_l)U(C)=\exp(-\delta\chi R_{lj}M_j)$ where $C=C(\delta\theta,n)$

Taking the second order approximation in $\delta\chi$ and after that in $\delta\theta$ it follows

$(\mathbb{I}+i\delta\theta n_kM_k)M_l(\mathbb{I}-i\delta\theta n_kM_k)=(\mathbb{I}-\delta\theta n_k\epsilon_{klj})M_j+O(\delta\theta^2)$ and so we will have the commutation relations:

$[M_k,M_l]=i\epsilon_{klj}M_j$ (41)

We take $X=M_1+iM_2$, $Y=M_1-iM_2$, $H=2M_3$ and we will have:

$[X,Y]=H$, $[H,X]=2X$, $[H,Y]=-2Y$ (42)

$M_1=N_1=\frac{1}{2}(X+Y)$, $M_2=N_2=\frac{1}{2}(iY-iX)$, $M_3=N_3=\frac{1}{2}H$

Suppose that *U* is finite-dimensional complex and irreducible.

Then exists an eigenvalue $\lambda\in\mathbb{C}$ of H with an eigenvector $v\in V$, $Hv=\lambda v$, $v\neq 0$

From $[H,X]=2X$ follows $HX^jv=(\lambda+2j)X^jv$ and the space being finite-dimensional we can take $i_0=\max\{i\in\mathbb{N}|X^iv\neq 0\}$. Let $v_0=X^{i0}v$, $v_j=Y^jv_0$.

From $[H,Y]=-2Y$ follows $Hv_j=(\lambda+2(i_0-j))v_j$ and the space being finite-dimensional we can take $m=\max\{i\in\mathbb{N}|v_i\neq 0\}$

We have

$X v_0 = 0$, $X v_{j+1} = X Y v_j = Y X v_j + H v_j = Y X v_j + (\lambda + 2(i_0 - j)) v_j$, $Y v_j = v_{j+1}, Y v_m = 0$

v_0, v_1, … v_m are linearly independent being eigenvectors of *H* for distinct eigenvalues and by induction follows from the above relations that *H*, *X*, *Y* leave invariant the subspace generated by them. The representation being irreducible, that subspace must be the whole space and *H* has therefore one-dimensional eigenspaces for each eigenvalue $\lambda + 2(i_0 - j)$, $j = \overline{0,m}$ with eigenvectors respective v_j . Therefore for the trace of *H* we have:

$$tr H = \sum_{j=0}^{m} (\lambda + 2(i_0 - j)) = (m+1)(\lambda + 2 i_0 - m) \ .$$

Since $tr H = tr[X, Y] = 0$ it follows $\lambda = m - 2 i_0$

By induction we can prove $X v_j = j(m - j + 1) v_{j-1}$ for $j = \overline{1,m}$ having $X v_0 = 0$.

In conclusion we will have $V = Sp[v_0, v_1, \ldots, v_m]$, $H v_j = (m - 2j) v_j$ for $j = \overline{0,m}$ and also $Y v_m = 0$, $Y v_j = v_{j+1}$ for $j = \overline{0, m-1}$ for the spin m/2 irreducible representation representation.

It can be proved without difficulties that if the *V* is the subspace of complex polynomials given by

$$V_m = \left\{ \sum_{j=0}^{m} a_j x^{m-j} y^j \in P[x,y] \middle| a_j \in \mathbb{C} \text{ for } j = \overline{0,m} \right\}$$

then $U : SL(2, \mathbb{C}) \to GL(V_m)$ with $U(A) p(x,y) = p(A^{-1}(x,y))$ for any $A \in SL(2, \mathbb{C})$ and any $p(x,y) \in V_m$, A^{-1} acting on the column vector (x, y) , provides a m+1-dimensional irreducible representation of $SL(2, \mathbb{C})$

For $A = \exp(-i \frac{1}{2} \theta \sigma_3)$ we have $A^{-1} = \cos(\frac{\theta}{2}) \mathbf{I} + i \sin(\frac{\theta}{2}) \sigma_3$ and

$$\exp(-i \frac{1}{2} \theta H)(x^{m-j} y^j) = U(A)(x^{m-j} y^j) =$$

$$= (\cos(\frac{\theta}{2}) + i \sin(\frac{\theta}{2}))^{m-j} (\cos(\frac{\theta}{2}) - i \sin(\frac{\theta}{2}))^j x^{m-j} y^j = \exp(i \frac{m-2j}{2} \theta) x^{m-j} y^j$$

Differentiating with respect to θ and taking $\theta = 0$ we obtain $H(x^{m-j} y^j) = (m - 2(m-j)) x^{m-j} y^j$ and so we have obtained the eigenvalues and eigenvectors of *H* in the representation.