On the rotation groups and the restricted Lorentz group

Consider the n-dimensional rotations group
$SO(n)=\{R\in M_{n\times n}(\mathbb{R}) | R R^T=\mathbf{I}\ ,\ det\, R=1\}$.
On the rotations group we take the topology induced from the square real matrices space such that a fundamental system of neighbourhoods of $R_0\in SO(n)$ is $(V_\varepsilon(R_0))_{\varepsilon>0}$ with
$V_\varepsilon(R_0)=\{R\in SO(n) \| R_{ij}-R_{0ij}|<\varepsilon$ for $i,j=\overline{1,n}\}$
For any $R=(Q_{ij})_{i,j}\in SO(n+1)$ if $Q_{k1}\neq 0$ we consider

$e_1^k=(Q_{i1})_{i=\overline{1,n+1}}$, $e_j^k=(\delta_{j-1\,i})_{i=\overline{1,n+1}}$ for $j=\overline{2,k}$, $e_j^k=(\delta_{ji})_{i=\overline{1,n+1}}$ for $j=\overline{k+1,n+1}$ and
$f_1^k=e_1^k$
$f_{p+1}^k=\text{vers}(e_{p+1}^k-\sum_{j=1}^{p}\langle f_j^k,e_{p+1}^k\rangle f_j^k)$ for $p=\overline{1,n-1}$

$f_{n+1}^k=\text{sign}(Q_{k1})\,\text{vers}(e_{n+1}^k-\sum_{j=1}^{n}\langle f_j^k,e_{n+1}^k\rangle f_j^k)$
Then if $Q_{k1}\neq 0$ for $Q^{(k)}=(f_{ij}^k)_{i,j=\overline{1,n+1}}$ we have $Q^{(k)}\in SO(n+1)$ and
$Q^{(k)}R=\begin{pmatrix}1 & 0_{1\times n}\\ 0_{n\times 1} & R^{(k)}\end{pmatrix}=M^{(k)}$ with
$R^{(k)}\in SO(n)$
where $\langle .,.\rangle$ denotes the euclidean scalar product and δ the Kronecker symbol
We suppose as a induction hypothesis that we have a C^∞ class mapping with
$W\ni(\psi_j)_{j=\overline{1,n(n-1)/2}}\rightarrow \overline{R}((\psi_j)_j)\in SO(n)$
W an open set of $\mathbb{R}^{n(n-1)/2}$ and rank $\left(\frac{\partial \overline{R}_{pq}}{\partial \psi_j}\right)_{pq,j}=\frac{n(n-1)}{2}$, $p,q=\overline{1,n}$, $j=\overline{1,n(n-1)/2}$
We take $(\psi_s)_{s=\overline{n(n-1)/2+1,n(n+1)/2}}\rightarrow(Q_{s1})_{s=\overline{1,n+1}}\in S_n$ where $S_n=\{x\in\mathbb{R}^{n+1} \| x\|=1\}$ is the n+1- dimensional sphere, a mapping of S_n.
We have $Q^{(k)T}=\begin{pmatrix}Q_{11} & A\\ B & C\end{pmatrix}$ with $A\in M_{1\times n}(\mathbb{R})$, $B\in M_{n\times 1}(\mathbb{R})$, $C\in M_{n\times n}(\mathbb{R})$.
$B=\begin{pmatrix}Q_{21}\\ \cdot\\ \cdot\\ Q_{n+11}\end{pmatrix}$ and $\begin{pmatrix}A\\ C\end{pmatrix}$ has an inverse $S\in M_{(n+1)\times n}(\mathbb{R})$: $S\begin{pmatrix}A\\ C\end{pmatrix}=\mathbf{I_n}$
It follows
$Q^{(k)T}M^{(k)}=\begin{pmatrix}Q_{11} & AR^{(k)}\\ B & CR^{(k)}\end{pmatrix}$ (1) and we can consider now a mapping $\overline{R}^{(k)}$ such that
$\overline{R}^{(k)}((\psi_s)_{s=\overline{1,n(n+1)/2}})=Q^{(k)T}\begin{pmatrix}1 & 0_{1\times n}\\ 0_{n\times 1} & \overline{R}\end{pmatrix}$
$Q^{(k)}=Q^{(k)}((\psi_s)_{s=\overline{n(n-1)/2+1,n(n+1)/2}})$
$\overline{R}=\overline{R}((\psi_j)_{j=\overline{1,n(n-1)/2}})$
Since rank $\left(\frac{\partial Q_{i1}}{\partial \psi_s}\right)_{i,s}=n$ with $i=\overline{1,n+1}$, $s=\overline{n(n-1)/2+1,n(n+1)/2}$
and the mapping $\overline{R}$ has rank $n(n-1)/2$ and we have the inverse S for

$\begin{pmatrix} A \\ C \end{pmatrix}$ it follows that if $\sum_{j=1}^{n(n+1)/2} \alpha_j \frac{\partial(Q^{(k)} M^{(k)})}{\partial \psi_j} = 0$ with $\alpha_j \in \mathbb{R}$, $j = \overline{1, n(n+1)/2}$

then $\alpha_j = 0$ for $j = \overline{1, n(n+1)/2}$ and so $\operatorname{rank}\left(\frac{\partial R_{pq}^{(k)}}{\partial \psi_j}\right)_{pq,j} = \frac{n(n+1)}{2}$

where $p, q = \overline{1, n+1}$, $j = \overline{1, n(n+1)/2}$

The application $\Phi^{(k)}: S_n \times SO(n) \to SO(n+1)$

$\Phi^{(k)}((Q_{i1})_{i=\overline{1,n+1}}, R^{(k)}) = M^{(k)}$

is a local homeomorphism, defined in the neighbourhood of each $((Q_{i1})_{i=\overline{1,n+1}}, R^{(k)})$ with $Q_{k1} \neq 0$ and so by induction we can define a smooth class C^∞ manifold structure on $SO(n+1)$ having dimension $n(n+1)/2$ that generates the same topology on $SO(n+1)$ as induced from $M_{(n+1)\times(n+1)}(\mathbb{R})$.

Suppose now the induction assumption that for any $R \in SO(m)$, $m \leq n$ exists $W \in M_{m \times m}(\mathbb{R})$ such that $W = -W^T$ and $R = \exp(W)$

For $Q \in SO(n+1)$ if Q invariates a subspace V of $\mathbb{R}^{n+1}$ (i.e. $Q(V) = V$)

then Q invariates also $V^\perp$

If $\lambda \in \mathbb{C}$, $x \in M_{(n+1)\times 1}(\mathbb{C})$, $x \neq 0$, $Qx = \lambda x$ it follows $\lambda \bar{\lambda} = 1$, since $Q Q^T = \mathbf{I}$

($\bar{\lambda}$ the complex conjugate of λ)

If further $\lambda \in \mathbb{C} \setminus \mathbb{R}$ taking $u = \Re x$, $v = \Im x$ we have

$\lambda = \cos(\theta) + i \sin(\theta)$, $\sin(\theta) \neq 0$, $v \neq 0$

$Qu = u\cos(\theta) - v\sin(\theta)$, $Qv = u\sin(\theta) + v\cos(\theta)$ and since $QQ^T = \mathbf{I}$ we will have

$$\|u\|^2 = \|u\|^2 \cos^2(\theta) + \|v\|^2 \sin^2(\theta) - 2\langle u, v\rangle \sin(\theta)\cos(\theta)$$

$$\|v\|^2 = \|u\|^2 \sin^2(\theta) + \|v\|^2 \cos^2(\theta) + 2\langle u, v\rangle \sin(\theta)\cos(\theta)$$

$$\langle u, v\rangle = (\|u\|^2 - \|v\|^2)\sin(\theta)\cos(\theta) + \langle u, v\rangle(\cos^2(\theta) - \sin^2(\theta))$$

$$(\|u\|^2 - \|v\|^2)(\cos(2\theta) - 1) - 2\langle u, v\rangle \sin(2\theta) = 0$$

$$(\|u\|^2 - \|v\|^2)\sin(2\theta) + 2\langle u, v\rangle(\cos(2\theta) - 1) = 0$$

Therefore, since $\sin(\theta) \neq 0$ it follows $\langle u, v\rangle = 0$, $\|u\| = \|v\| \neq 0$

Q invariates $\operatorname{Sp}(u, v)$ and $\operatorname{Sp}(u, v)^\perp$ and so we can find $R \in SO(n+1)$ such that

$$RQR^T = \begin{pmatrix} B & 0_{2\times(n-1)} \\ 0_{(n-1)\times 2} & Q_0 \end{pmatrix}$$

with $Q_0 = \exp(W_0) \in SO(n-1)$, $W_0 = -W_0^T \in M_{(n-1)\times(n-1)}(\mathbb{R})$

$B = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \exp(\theta A)$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Taking $W = R^T \begin{pmatrix} \theta A & 0_{2\times(n-1)} \\ 0_{(n-1)\times 2} & W_0 \end{pmatrix} R$ we obtain $Q = \exp(W)$

If $\lambda \in \mathbb{R}$ we must have $\lambda \in \{-1, 1\}$

Then if $x \in M_{(n+1)\times 1}(\mathbb{R})$, $Qx = -x$, $x \neq 0$ since $\det Q = 1$ there it exists another $y \neq 0$, $y \in \{x\}^\perp$ such that $Qy = -y$ and we will have

$R \in SO(n-1)$ with $RQR^T = \begin{pmatrix} \exp(\pi A) & 0_{2\times(n-1)} \\ 0_{(n-1)\times 2} & Q_0 \end{pmatrix}$, $Q_0 = \exp(W_0)$, $W_0 = -W_0^T$

and we can take $Q = \exp(W)$ with $W = R^T \begin{pmatrix} \pi A & 0_{2\times(n-1)} \\ 0_{(n-1)\times 2} & W_0 \end{pmatrix} R = -W^T \in M_{(n+1)\times(n+1)}(\mathbb{R})$

If $x \in M_{(n+1)\times 1}(\mathbb{R})$, $x \neq 0$, $Qx = x$ we will have $R \in SO(n+1)$ such that $RQR^T = \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & Q_0 \end{pmatrix}$ where by induction assumption $Q_0 = \exp(W_0), W_0 = -W_0^T$

Taking $W = R^T \begin{pmatrix} 0 & 0_{1\times n} \\ 0_{n\times 1} & W_0 \end{pmatrix} R = -W^T \in M_{(n+1)\times(n+1)}(\mathbb{R})$ we will have $Q = \exp(W)$

Therefore , by induction we have proved that for any $Q \in SO(n)$ exists $W \in M_{n\times n}(\mathbb{R})$ such that $W = -W^T$ and $Q = \exp(W)$

Also it is obvious that if $W = -W^T$ then $WW^T = W^T W$ and so for $Q = \exp(W)$: $QQ^T = \exp(W + W^T) = \mathbf{I}$, $Q \in O(n)$

Moreover we have $W = JSJ^{-1}$ where S is the Jordan normal form of W

$\det \exp(W) = \det \exp(S) = \prod_{i=1}^{n} \exp(\lambda_i)$, where

$\det(W - \lambda \mathbf{I}) = \prod_{i=1}^{n} (\lambda - \lambda_i)$

For $W = -W^T \in M_{n\times n}(\mathbb{R})$, $Wx = \lambda x$, $x \in M_{n\times 1}(\mathbb{C})$, $\lambda \in \mathbb{R}$ we can take $x \in M_{n\times 1}(\mathbb{R})$ and so $x^T W x = x^T W^T x = -x^T W x = 0$, $0 = x^T W x = \lambda \|x\|^2$ and all real eigenvalues of *W* must vanish and since *W* is real we can split the eigenvalues as

$E = \{i \in \{1, ..n\} | \lambda_i \in \mathbb{C} \setminus \mathbb{R}\} = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$, $\operatorname{card} E_1 = \operatorname{card} E_2$

$E_1 = \{i_1, ..., i_k\}$, $E_2 = \{j_1, ..., j_k\}$, $\lambda_{is} = \overline{\lambda}_{js}$ for $s = \overline{1, k}$

Therefore it follows $\prod_{i=1}^{n} \exp(\lambda_i) > 0$, $Q = \exp(W) \in SO(n)$.

We will prove now that the function $\Phi : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ with $\Phi(W) = \exp(W)$ for any $W \in M_{n\times n}(\mathbb{R})$ satisfies

$$\det\left(\frac{\partial \Phi_{pq}}{\partial t_{ij}}\right)_{pq,ij} \neq 0 \text{ for any } W = (t_{ij})_{i,j} \in M_{n\times n}(\mathbb{R}) \text{ where } i, j, p, q = \overline{1, n} \qquad (2)$$

$\Phi = (\Phi_{pq})_{p,q} = (\mathrm{e}^W_{pq})_{p,q}$

(2) is equivalent to the fact that

for any W , $\beta \in M_{n\times n}(\mathbb{R})$ the relation $\frac{\partial}{\partial h} \exp(W + h\beta)¿\Big|_{h=0} = 0$ implies $\beta = 0$.

Suppose we have $W, \beta \in M_{n\times n}(\mathbb{R})$ such that $\frac{\partial}{\partial h} \exp(W + h\beta)\Big|_{h=0} = 0$

Since for any $J \in M_{n\times n}(\mathbb{C})$ with $\det J \neq 0$ we have $\frac{\partial}{\partial h} \exp(JWJ^{-1} + hJ\beta J^{-1}) = J\left(\frac{\partial}{\partial h} \exp(W + h\beta)\right) J^{-1}$ it is sufficient to prove (2) only for *W* having the upper triangular normal Jordan form

$$W = \begin{pmatrix} C_1 & 0 & .. & 0 \\ 0 & C_2 & .. & 0 \\ . & .. & . & 0 \\ 0 & .. & 0 & C_r \end{pmatrix}$$ with C_i cells of the form $C_i = \lambda_i \mathbf{I}_{si} + N_{si}$, $\lambda_i \in \mathbb{C}$, $s_i \in \mathbb{N}^*$

$N_{si} = (n_{kl})_{k,l=\overline{1,si}}$, $n_{kl} = \begin{cases} 1 & \text{if } l = k+1, 1 \leq k \leq s_i - 1 \\ 0 & \text{otherwise} \end{cases}$ for $k, l = \overline{1, s_i}$

We denote $W_{ii} = \mu_i$ for $i = \overline{1, n}$

Let *W* having the Jordan normal form and we have :

$$\frac{\partial e^W}{\partial t_{ij}}=\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{k=0}^{m-1}W^{m-1-k}E_{ij}W^k \text{ where } E_{ij}=(\delta_{ik}\delta_{jl})_{k,l=\overline{1,n}} \qquad (3)$$

For any $k\in\mathbb{N}$, W^k is upper triangular and has diagonal coefficients $(W^k)_{ii}=\mu_i^k$ and therefore calculating the terms $W^{m-1-k}E_{ij}W^k$ it follows that $\frac{\partial e^W_{pq}}{\partial t_{ij}}=0$ if $q<j$ or $p>i$.

We consider for pq and ij pass the ordering (1n)(2n))...((n-1)n)(nn)(1(n-1))...(n(n-1))(1(n-2))…...(11)((21)...((n-1)1)(n1) we find that the matrix

$\left(\frac{\partial e^W_{pq}}{\partial t_{ij}}\right)_{pq,ij}$ has an upper triangular form and so

$$\det\left(\frac{\partial e^W_{pq}}{\partial t_{ij}}\right)_{pq,ij}=\prod_{p,q=1}^{n}\frac{\partial e^W_{pq}}{\partial t_{pq}}$$

Calculation from (3) , with W having the Jordan normal form leads to

$$\frac{\partial e^W_{pq}}{\partial t_{pq}}=\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{k=0}^{m-1}\mu_p^{m-1-k}\mu_q^k=\begin{cases}\dfrac{\exp(\mu_p)-\exp(\mu_q)}{\mu_p-\mu_q} & \text{if } \mu_p\neq\mu_q\\ \dfrac{\exp(\mu_p)-1}{\mu_p} & \text{if } \mu_p=\mu_q\end{cases}$$

and so $\det\left(\frac{\partial e^W_{pq}}{\partial t_{ij}}\right)_{pq,ij}\neq 0$ for any $W\in M_{n\times n}(\mathbb{R})$ (3’)

Let $(J_s)_{s=\overline{1,n(n-1)/2}}$ a system of linear independent generators for the antisymmetric real matrices so that we have

$W=-W^T\in M_{n\times n}(\mathbb{R})W=\psi_s J_s$, $\psi_s\in\mathbb{R}$ (with Einstein summation convention for indexes $s=\overline{1,n(n-1)/2}$)

Because we have (3’), it follows that for

$R_0\in SO(n)$ and $\psi^0=(\psi_s^0)_s$ such that $\exp(\psi_s^0 J_s)=R_0$ we have an open neighbourhood U_0 of ψ^0 , an open neighbourhood G_0 of R_0 and the injective function $\Phi:U_0\to G_0$, $\Phi(\psi)=\exp(\psi_s J_s)$, $\psi=(\psi_s)_s$

As we proved , we can choose G_0 such that we have a mapping

$\bar{R}:V_0\to G_0$ of $SO(n)$ from some open neighbourhood V_0 of φ^0 such that $\varphi^0=(\varphi_s^0)_s$ and $\bar{R}(\varphi^0)=R_0$, $\operatorname{rank}\left(\frac{\partial\bar{R}_{pq}}{\partial\varphi_s}\right)_{pq,s}=\frac{n(n-1)}{2}$ (4)

Thus we have $\bar{R}(\varphi^0)=\exp(\psi_s^0 J_s)$ and for $\varphi=\bar{R}^{-1}\circ\Phi(\psi)$ we have $\bar{R}(\varphi)=\exp(\psi_s J_s)$ for any $\psi\in U_0$ and the function $\bar{R}^{-1}\circ\Phi:U_0\to V_0$ is continuous and injective .

Therefore, since U_0 is open and U_0 , V_0 have the same dimension it follows that $\bar{R}^{-1}\circ\Phi(U_0)=W_0$ is an open set and $\varphi^0\in W_0$ and we have a homeomorphism $\bar{R}^{-1}\circ\Phi:U_0\to W_0$

Since (4) , by the implicit function theorem we will have a C^1 class function $h:U_1\to V_1$ with U_1 open neighbourhood of ψ^0 , V_1 open neighbourhood of φ^0 such that $\bar{R}(h(\psi))=\exp(\psi_s J_s)$ for any $\psi\in U_1$

and for any $(\varphi,\psi)\in V_1\times U_1$: $\bar{R}(\varphi)=\exp(\psi_s J_s)$ if and only if $\varphi=h(\psi)$

Since (3’) we have that

rank$\left(\frac{\partial \exp(\psi_s J_s)_{pq}}{\partial \psi_k}\right)_{pq,k}=\frac{n(n-1)}{2}$ and we will have also a C[1] class function

$g: V_2 \to U_2$ with U_2 open neighbourhood of ψ^0 , V_2 open neighbourhood of φ^0 such that $\bar{R}(\varphi)=\exp(g_s(\varphi) J_s)$ for any $\varphi \in V_2$

and for any $(\varphi, \psi) \in V_2 \times U_2 : \bar{R}(\varphi)=\exp(\psi_s J_s)$ if and only if $\psi = g(\varphi)$.

It follows $(g \circ h)(\psi)=\psi$, $(h \circ g)(\varphi)=\varphi$ for any $(\varphi, \psi) \in V_1 \cap V_2 \times U_1 \cap U_2$

Therefore we can find *U*, *V* open neighbourhoods of ψ^0 respective φ^0 such that

$h(U)=V$, $g(V)=U$, $h|_U = g^{-1}|_U$, $\bar{R}(h(\psi))=\exp(\psi_s J_s)$, $\bar{R}(\varphi)=\exp(g_s(\varphi) J_s)$ for any $(\varphi, \psi) \in V \times U$.

Intermediating through the $\bar{R}$ mappings of the manifold structure $SO(n)$ we obtain that for any ψ^0 , ψ^1 with $\exp(\psi^0_s J_s)=\exp(\psi^1_s J_s)$ there exist W_0 an open neighbourhood of ψ^0 and W_1 an open neighbourhood of ψ^1 and a C^∞ class function $f: W_0 \to W_1$ such that for any $(\psi, \psi') \in W_0 \times W_1$:

$\exp(\psi_s J_s)=\exp(\psi'_s J_s)$ if and only if $\psi' = f(\psi)$ and so we have the same manifold structure on $SO(n)$ with topology induced from $M_{n\times n}(\mathbb{R})$ given by the mappings $(\psi_s)_{s=\overline{1,n(n-1)/2}} \to \exp(\psi_s J_s)$ having the continuous surjective function $\Phi: \mathbb{R}^{n(n-1)/2} \to SO(n)$ with $\Phi(\psi)=\exp(\psi_s J_s)$ and so we find $SO(n)$ as a $n(n-1)/2$ - dimensional connected Lie group.

Consider now the Minkowski space

$\mathbb{R}^4$ identified with $M_{4\times 1}(\mathbb{R})$ having the pseudometric $(\eta_{\alpha\beta})_{\alpha,\beta}$ with

$\eta_{\alpha\beta}=0$ if $\alpha \neq \beta$, $\eta_{ii}=-1$ for $i=\overline{1,3}$, $\eta_{00}=1$

(we use greek characters for indexing from 0 to 3 and latin characters for indexing from 1 to 3)

We have the pseudo-scalar product $\mathbb{R}^4 \times \mathbb{R}^4 \ni (x, y) \to x \cdot y = y^T \eta x \in \mathbb{R}$

x , y as column vectors $x = x^\alpha E_\alpha$, $(E_\alpha)_\alpha$ Minkowski base with

$E_\alpha = (\delta_{\alpha\beta})_\beta$ (as column vector) , $E_\alpha \cdot E_\beta = \eta_{\alpha\beta}$

We remind that , as a consequence of the Cauchy-Bunyakowsky-Schwarz inequality, we have:

i) if $x, y \in \mathbb{R}^4$ and $x \neq 0$, $x^T \eta x \geq 0$, $y^T \eta x = 0$ then $y^T \eta y \leq 0$

ii) if x , $y \in \mathbb{R}^4$ and $x \neq 0$, $x^T \eta x = 0$, $y^T \eta x = 0$ then exists $\lambda \in \mathbb{R}$ with $y = \lambda x$.

For $M \in SO^+(3,1)$ (see Chap. Representations of the rotations group and of the restricted Lorentz group , Spin representations) we have :

$M = R(\theta, n) B(\chi, q) = M(\vec{\theta}, \vec{\chi})$ where $\vec{\theta} = \theta n$, $\vec{\chi} = \chi q$, $\vec{\theta} = (\theta_i)_i$, $\vec{\chi} = (\chi_i)_i$

$R(\theta, n) = (R_{\alpha\beta})_{\alpha,\beta}$, $B(\chi, q) = (B_{\alpha\beta})_{\alpha,\beta}$,

$R_{ij} = -\epsilon_{ijk} n_k \sin(\theta) + (\delta_{ij} - n_i n_j)\cos(\theta) + n_i n_j$, $R_{i0} = R_{0i} = 0$, $R_{00} = 1$

$B_{ij} = \delta_{ij} + (\cosh(\chi) - 1) q_i q_j$, $B_{0i} = B_{i0} = -q_i \sinh(\chi)$, $B_{00} = \cosh(\chi)$

B is symmetric positive definite and so $M = RB$ must be the polar decomposition of M ,

$B = \sqrt{M^T M}$, $R = M(\sqrt{M^T M})^{-1}$ and we can find $k \in \{1,2,3\}$ such that :

$n = \mathrm{vers}(\epsilon_{ijk}(R_i - \delta_i) \times (R_j - \delta_j))$ with $R_i = (R_{il})_l$, $\delta_i = (\delta_{il})_l$

$\sin(\theta) = -\frac{1}{2} \epsilon_{ijl} n_l R_{ij}$, $\cos(\theta) = \frac{1}{2}(R_{ii} - 1)$

$\cosh(\chi) = B_{00}$, $\sinh(\chi) = \sqrt{B_{00}^2 - 1}$, $q_i = -\frac{B_{i0}}{\sqrt{B_{00}^2 - 1}}$

Therefore we have a local homeomorphism :

$\mathbb{R}^6 \ni (\vec{\theta},\vec{\chi}) \to M(\vec{\theta},\vec{\chi}) \in SO^+(3,1)$ when $SO^+(3,1)$ is considered with the topology which is induced from $M_{4\times 4}(\mathbb{R})$ and a 6-dimensional connected Lie group structure on $SO^+(3,1)$ given by the mappings $(\vec{\theta},\vec{\chi}) \to M(\vec{\theta},\vec{\chi})$.

Suppose now we have

$(\alpha_k)_k$, $(\beta_k)_k \in \mathbb{R}^3$ such that $\alpha_k \dfrac{\partial M}{\partial \theta_k} + \beta_k \dfrac{\partial M}{\partial \chi_k} = 0$ for a value of $(\vec{\theta},\vec{\chi})$

It follows :

$$0 = \beta_k \frac{\partial B_{0i}}{\partial \chi_k} = \beta_k \left(-\frac{\delta_{ik}}{\chi} + \frac{1}{\chi} q_i q_k \right) \sinh(\chi) - \beta_k q_k q_i \cosh(\chi)$$

$0 = \beta_k \dfrac{\partial B_{00}}{\partial \chi_k} = \beta_k q_k \sinh(\chi)$ and so we obtain $\beta_i = 0$ for $i = \overline{1,3}$

$\alpha_k \dfrac{\partial R}{\partial \theta_k} = 0$ with $(R_{ij})_{i,j} = \exp(\theta_k J_k)$, $(J_k)_{ij} = -\epsilon_{ijk}$, for $i,j,k = \overline{1,3}$.

Since $\det \left(\dfrac{\partial e^W_{pq}}{\partial t_{ij}} \right)_{pq,ij} \neq 0$ for any $W = (t_{ij})_{i,j} \in M_{3\times 3}(\mathbb{R})$ as we have proven, it follows :

$\operatorname{rank} \left(\dfrac{\partial \exp(\theta_k J_k)_{ij}}{\partial \theta_l} \right)_{ij,l} = 3$ and so we must have also $\alpha_k = 0$ for $k = \overline{1,3}$.

Therefore taking $(\psi_l)_{l=\overline{1,6}} = (\vec{\theta},\vec{\chi})$ we have $\operatorname{rank} \left(\dfrac{\partial M_{\alpha\beta}}{\partial \psi_l} \right)_{\alpha\beta,l} = 6$

We remind that we rise or lower the indexes according to

$V_\alpha = \eta_{\alpha\beta} V^\beta$, $V^\alpha = \eta^{\alpha\beta} V_\beta$, $(\eta_{\alpha\beta}) = (\eta^{\alpha\beta})$

Let $\epsilon_{\alpha\beta\gamma\delta}$ be the signature of the permutation $\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & 1 & 2 & 3 \end{pmatrix}$ and we define

$\bar{J}^{\alpha\beta}_{\gamma\delta} = \epsilon^{\alpha\beta\gamma\varepsilon} \eta_{\varepsilon\delta}$. We will have:

$\bar{J}^{0i} = -\bar{J}^{i0} = -J_i$, $\bar{J}^{ij} = \epsilon_{ijk} K_k$ where J_i , K_k are the Lorentz group generators

$(J_k)_{ij} = -\epsilon_{ijk}$, $(J_k)_{i0} = (J_k)_{0i} = (J_k)_{00} = 0$, $(K_k)_{ij} = 0$, $(K_k)_{i0} = (K_k)_{0i} = \delta_{i0}$, $(K_k)_{00} = 0$

$R(\theta,n) = \exp(\theta n_k J_k)$, $B(\chi,q) = \exp(-\chi q_k K_k)$.

We define also

$J^{\alpha\beta}_{\gamma\delta} = \dfrac{1}{2} \epsilon^{\alpha\beta}{}_{\psi\varphi} \bar{J}^{\psi\varphi}_{\gamma\delta} = -\dfrac{1}{2} \epsilon^{\alpha\beta\psi\varphi} \epsilon_{\psi\varphi\delta\rho} \eta^{\gamma\rho}$ obtaining

$J^{ij} = -\epsilon_{ijk} J_k$, $J^{0i} = -J^{i0} = -K_i$.

For a Lorentz coordinates transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, $(\Lambda^\mu{}_\nu)_{\mu,\nu} = \Lambda \in SO^+(3,1)$ we denote $(\Lambda_\mu{}^\nu)_{\nu,\mu} = \Lambda^{-1}$.

The relation $\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \bar{J}^{\alpha\beta} = \Lambda^{-1} \bar{J}^{\mu\nu} \Lambda$ (5) is equivalent to

$\Lambda^\varepsilon{}_\gamma \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda_\rho{}^\delta \bar{J}^{\alpha\beta}_{\gamma\delta} = \bar{J}^{\mu\nu}_{\varepsilon\rho}$.

Since $\Lambda \in SO^+(3,1)$ we have $\Lambda_\rho{}^\delta = \Lambda^\varphi{}_\psi \eta_{\psi\delta} \eta_{\rho\varphi}$ and so (5) is equivalent to

$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\varepsilon{}_\gamma \Lambda^\kappa{}_\psi \epsilon^{\alpha\beta\gamma\psi} = \epsilon^{\mu\nu\varepsilon\kappa}$ which is true since $\det \Lambda = 1$

Also, since $\Lambda \in SO^+(3,1)$ we have

$\epsilon^{\gamma\varepsilon}{}_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta^{\kappa\gamma} \eta^{\rho\varepsilon} \epsilon_{\alpha\beta\psi\varphi} \Lambda_\kappa{}^\psi \Lambda_\rho{}^\varphi = \epsilon^{\kappa\rho}{}_{\alpha\beta} \Lambda^\gamma{}_\kappa \Lambda^\varepsilon{}_\rho$.

Therefore, from (5) follows :

$\Lambda^\nu{}_\alpha \Lambda^\nu{}_\beta J^{\alpha\beta} = \Lambda^{-1} J^{\mu\nu} \Lambda$ (6)

We will prove further that if $M\in SO^+(3,1)$ then exists $\Lambda\in SO^+(3,1)$ such that $M=\Lambda^{-1}\exp(\theta J_3+\chi K_3)\Lambda$ or $M=\Lambda^{-1}\exp(\alpha(J_1+K_2))\Lambda$ for some $\theta,\chi,\alpha\in\mathbb{R}$.

If exists $\mu\in\mathbb{C}\setminus\mathbb{R}$ such that we have $x\in\mathbb{C}^4$(x as a column vector) $x\neq 0$ with $Mx=\mu x$, then if $x^T\eta x\neq 0$ it follows, since $M^T\eta M=\eta$ that $\mu^2=1$ and because $\mu\in\mathbb{C}\setminus\mathbb{R}$, we must have $x^T\eta x=0$ and so

$\Re x^T\eta\Re x=\Im x^T\eta\Im x$, $\Re x^T\eta\Im x=0$ (7)

If also $\Re x^T\eta\Re x=0$, since $\Im x\neq 0$(because $\mu\notin\mathbb{R}$) we have $\lambda\in\mathbb{R}$ with $\Re x=\lambda\Im x$.

This leads to $(i+\lambda)M\Im x=(i+\lambda)\mu\Im x$ which again contradicts $\mu\notin\mathbb{R}$.

Therefore we have $u=\Re x$, $v=\Im x$, $x=u+iv$

$u^T\eta u=v^T\eta v\neq 0$, $u^T\eta v=0$ (8)

$\bar{x}^T\eta x=u^T\eta u+v^T\eta v\neq 0$ (9) which from $Mx=\mu x$ leads to $\mu\bar{\mu}=1$ and $\alpha\in\mathbb{R}$ with

$Mu=u\cos(\alpha)-v\sin(\alpha)$

$Mv=u\sin(\alpha)+v\cos(\alpha)$

As a consequence of Cauchy-Bunyakowsky-Schwarz inequality, from (8) we obtain $u^T\eta u=v^T\eta v<0$ and we can therefore consider $u^T\eta u=v^T\eta v=-1$.

M invariates $V=\mathrm{Sp}(u,v)$ and for $V^\perp=\{w\in M_{4\times 1}(\mathbb{R})|w^T\eta z=0$ for any $z\in V\}$, M invariates also $V^\perp$.

We can take $\Lambda\in SO^+(3,1)$ such that $\Lambda^{-1}E_1=u$ and $\Lambda^{-1}E_2=v$.

For $\bar{M}=\Lambda M\Lambda^{-1}$ we will have:

$\bar{M}E_1=E_1\cos(\alpha)-E_{\sin}(\alpha)$, $\bar{M}E_2=E_1\sin(\alpha)+E_2\cos(\alpha)$ and that $\bar{M}$ invariates $\mathrm{Sp}(E_1,E_2)$ and $\mathrm{Sp}(E_3,E_0)=\mathrm{Sp}(E_1,E_2)^\perp$.

Hence exist θ , $\chi\in\mathbb{R}$ such that $\bar{M}=\exp(\theta J_3)\exp(\chi K_3)=\exp(\theta J_3+\chi K_3)$

$$M=\Lambda^{-1}\exp(\theta J_3+\chi K_3)\Lambda$$

Therefore, to prove the statement we can further suppose that

if $\mu\in\mathbb{C}$, $x\in M_{4\times 1}(\mathbb{C})$, $x\neq 0$, $Mx=\mu x$ then $\mu\in\mathbb{R}^*$ and $x\in M_{4\times 1}(\mathbb{R})$

Let $x\in M_{4\times 1}(\mathbb{R})$, $\lambda\in\mathbb{R}^*$, $x\neq 0$, $Mx=\lambda x$

If $x^T\eta x=0$ we can choose $x\in M_{4\times 1}(\mathbb{R})$ and take $\Lambda\in SO^+(3,1)$ such that $\Lambda x=E$ where $E=E_3+E_0$. Then for $\bar{M}=\Lambda M\Lambda^{-1}$, $\bar{M}$ invariates $\{E\}^\perp=\mathrm{Sp}(E_1,E_2,E)=V$ and we will have:

$\bar{M}E_1=\alpha E_1+\beta E_2+\gamma E$

$\bar{M}E_2=\alpha' E_1+\beta' E_2+\gamma' E$ (10)

$\bar{M}E=\lambda E$

Since $\bar{M}\in SO^+(3,1)$ we obtain:

$\alpha^2+\beta^2=1$, $\alpha'^2+\beta'^2=1$, $\alpha\alpha'+\beta\beta'=0$

$\alpha=\cos(\theta)$, $\beta=\sin(\theta)$, $\alpha'=\cos(\theta')$, $\beta'=\sin(\theta')$, $\theta-\theta'=\frac{2k+1}{2}\pi$, $k\in\mathbb{Z}$.

Let $S=\begin{pmatrix}\alpha & \alpha' & 0\\ \beta & \beta' & 0\\ \gamma & \gamma' & \lambda\end{pmatrix}$

After some calculus we find that solutions for the characteristic equation in μ are:

$\mu=\lambda$ and $\mu=\frac{1}{2}(1+(-1)^{k+1}\pm\sqrt{(1+(-1)^{k+1})\cos^2(\theta)+4(-1)^k})$

If $k\equiv 1(\mathrm{mod}\,2)$ and $\cos^2(\theta)\neq 1$, S and therefore also M has an eigenvalue which is not real and so we can consider that $k\equiv 0(\mathrm{mod}\,2)$ if $\cos^2(\theta)\neq 1$.

If $k\equiv 0(\mathrm{mod}\,2)$ or $\cos^2(\theta)\neq 1$, $\overline{M}$ must have an eigenvalue $\mu\in\mathbb{R}^*$, $\mu\neq\lambda$, $\mu^2=1$.
$\overline{M}y=\mu y$, $\overline{M}E=\lambda E$, $y\in V=\{E\}^{\perp}=\mathrm{Sp}(E_1,E_2,E)$.
$\overline{M}$ invariates $\mathrm{Sp}(y,E)$ and $\mathrm{Sp}(y,E)^{\perp}=W$ having $\dim W=2$.

For $z\in W$ we have $z^T\eta E=0$ and as a consequence of the Cauchy-Bunyakowsky-Schwarz inequality, for any $z\in W$ which is independent of E follows $z^T\eta z<0$.

Let $E'\in W$, $E'^T\eta E'=-1$ and we have β'' , $\alpha''\in\mathbb{R}$, $\beta''^2=1$ with

$$\overline{M}y=\mu y$$
$$\overline{M}E'=\beta''E'+\alpha''E \quad (10')$$
$$\overline{M}E=\lambda E$$

$y^T\eta E'=0$, $E'^T\eta E=0$, $E^T\eta y=0$, $y^T\eta y=-1$, $E'^T\eta E'=-1$, $E^T\eta E=0$ (10'')

$\mathrm{Sp}(y,E',E)=\mathrm{Sp}(E_1,E_2,E)$ and from (10) , (10') follows

$\det(S-\rho\mathbf{I})=\det(S'-\rho\mathbf{I})$ for any $\rho\in\mathbb{C}$ where $S'=\begin{pmatrix}\mu & 0 & 0\\ 0 & \beta'' & 0\\ 0 & \alpha'' & \lambda\end{pmatrix}$ and so we must have

$(\mu=1$ and $\beta''=-1)$ or $(\mu=-1$ and $\beta''=1)$.

Because $\det M=1$, the characteristic equation in ρ , $\det(M-\rho\mathbf{I})=0$ must have another solution $\rho=-\frac{1}{\lambda}$ and since $\lambda\in\mathbb{R}$ we have $\lambda\neq-\frac{1}{\lambda}$ and $z\in M_{4\times 1}(\mathbb{R})$, z independent of

x such that $Mz=-\frac{1}{\lambda}z$, $Mx=\lambda x$, $z^T\eta x=-z^T\eta x=0$.

Hence because $x^T\eta x=0$, x cannot be independent of z , and so, when all eigenvalues of M are real, as we can consider , we must suppose that we are in case a) or case b) described below:

a) for any $\lambda\in\mathbb{C}$, $x\in M_{4\times 1}(\mathbb{C})$, $x\neq 0$ with $Mx=\lambda x$ we can consider that
$\lambda\in\mathbb{R}^*$, $x\in M_{4\times 1}(\mathbb{R})$, $x\neq 0$, $Mx=\lambda x$, $x^T\eta x\neq 0$, $\lambda^2=1$

(the last equality in case a) follows because M is a Lorentz transformation)

b) there exist an eigenvalue $\lambda\in\mathbb{R}^*$ such that the corresponding x , θ , k which we have for λ satisfy $x^T\eta x=0$, $\cos^2(\theta)=1$, $k\equiv 1(\mathrm{mod}\,2)$.

In case a), taking $x_0\in M_{4\times 1}(\mathbb{R})$ with $x_0\neq 0,\lambda_0\in\mathbb{R}^*$, $Mx_0=\lambda_0x_0$ we have that M invariates $\{x_0\}$ and $\{x_0\}^{\perp}$ and we can take successively $x_i\in M_{4\times 1}(\mathbb{R})$, $x_i\neq 0$, $\lambda_i\in\mathbb{R}^*$ such that after eventually a permutation of indexes we have:

$Mx_\alpha=\lambda_\alpha x_\alpha$, $\lambda_\alpha^2=1$, $x_\alpha^T\eta x_\beta=\eta_{\alpha\beta}$ for α , $\beta=\overline{0,3}$.

Then we can find $\varepsilon_\alpha\in\{1,-1\}$, $\Lambda\in SO^+(3,1)$ with $\Lambda^{-1}E_\alpha=\varepsilon_\alpha x_\alpha$.

In the basis (E_1,E_2,E_3,E_0) the transformation $\overline{M}'=\Lambda M\Lambda^{-1}$ has the diagonal form

$\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_0\end{pmatrix}$ and since $\overline{M}'\in SO^+(3,1)$ we have $\lambda_0=1$.

If $\lambda_3=-1$ it follows $\lambda_1\lambda_2=-1$ and we can take $Q\in SO^+(3,1)$ with $Q^T=Q^{-1}$, $Q\overline{M}'Q^T=\exp(\pi J_3+0K_3)$. If $\lambda_3=1$ it follows $\lambda_1\lambda_2=1$ and also we can take $Q\in SO^+(3,1)$ with $Q^T=Q^{-1}$, $Q\overline{M}'Q^T\in\{\exp(\pi J_3+0K_3)\ ,\ \exp(0J_3+0K_3)\}$

In case b) we must have $\mu=\beta''=\cos(\theta)\in\{\pm 1\}$ and the characteristic equation has another solution $\rho=\frac{1}{\lambda}$, $\det(M-\rho\mathbf{I})=0$.

If $\beta'' \neq \lambda$, taking $z = E' + \frac{\alpha''}{\beta'' - \lambda} E$ we obtain:

$$\overline{M} y = \mu y \ , \ \overline{M} z = \mu z \ , \ \overline{M} E = \lambda E \ ,$$

$y^T \eta z = 0$, $y^T \eta y = z^T \eta z = -1$, $E \in \{y, z\}^\perp$ and we find $\Lambda \in SO^+(3,1)$ with $\varepsilon \in \{\pm 1\}$, $\rho \in \mathbb{R}^*$, $\Lambda^{-1} E_1 = y$, $\Lambda^{-1} E_2 = \varepsilon z$, $\Lambda^{-1} E = \rho E$.

For $M' = \Lambda \overline{M} \Lambda^{-1}$ we obtain $M' E_1 = \mu E_1$, $M' E_2 = \mu E_2$, $M' E = \lambda E$

Since $M' \in SO^+(3,1)$ it follows $\lambda > 0$, M' invariates $\mathrm{Sp}(E_1, E_2) = H$ and $\mathrm{Sp}(E_3, E_0) = H^\perp$

We will have therefore :

$M' = \exp(\theta J_3 + \chi K_3)$, $\theta \in \{0, \pi\}$, $\cosh(\chi) + \sinh(\chi) \in \left\{\lambda, \frac{1}{\lambda}\right\}$

If $\lambda^2 \neq 1$ we have obviously $\beta'' \neq \lambda$ and so we have now left the case $\lambda^2 = 1$, $\beta'' = \lambda$ having now the situation:

$\mu = \lambda = \beta'' = \cos(\theta) \in \{\pm 1\}$, $k \equiv 1 (\mathrm{mod}\, 2)$, $\sin(\theta) = \cos(\theta') = 0$, $\sin(\theta') = \cos(\theta)$

$\overline{M} E_1 = \mu E_1 + \gamma E$, $\overline{M} E_2 = \mu E_2 + \gamma' E$, $\overline{M} E = \mu E$ and so in the basis (E_1, E_2, E_3, E_0):

$\overline{M} = \begin{pmatrix} \mu & 0 & \delta & -\delta \\ 0 & \mu & \varepsilon & -\varepsilon \\ \gamma & \gamma' & \rho & \mu - \rho \\ \gamma & \gamma' & \varphi & \mu - \varphi \end{pmatrix}$ and $\overline{M} \in SO^+(3,1)$ leading to:

$\gamma = -\delta$, $\gamma' = -\varepsilon$, $\mu(\rho - \varphi) = 1$, $\mu(\delta^2 + \varepsilon^2) = -2\varphi$, $\mu \geq \varphi + 1$

If $\mu < 0$ it will follow $\varphi \geq 0$, $\mu \geq 1$ and so we must have $\mu = 1$, $\rho = \varphi + 1$, $\delta^2 + \varepsilon^2 = -2\varphi$

Taking $\overline{Q} = \begin{pmatrix} Q & 0_{2\times 2} \\ 0_{2\times 2} & \mathbf{I}_2 \end{pmatrix}$ with $Q = \begin{pmatrix} \cos(\zeta) & \sin(\zeta) \\ -\sin(\zeta) & \cos(\zeta) \end{pmatrix} \in SO(2)$ where $\zeta \in \mathbb{R}$, $\delta \cos(\zeta) + \varepsilon \sin(\zeta) = 0$ we have that $\overline{Q}\,\overline{M}\,\overline{Q}^T$ has the form:

$S(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha & -\alpha \\ 0 & -\alpha & 1 - \alpha^2/2 & \alpha^2/2 \\ 0 & -\alpha & -\alpha^2/2 & 1 + \alpha^2/2 \end{pmatrix}$ with $\alpha \in \mathbb{R}$.

After some calculus we find out that $S(\alpha + \alpha') = S(\alpha) S(\alpha')$ for any $\alpha, \alpha' \in \mathbb{R}$ and so

$\frac{dS}{d\alpha} = S \frac{dS}{d\alpha}(0) = -S(J_1 + K_2)$, $S(\alpha) = \exp(-\alpha(J_1 + K_2))$

Thus the statement is completely proved :

For any $M \in SO^+(3,1)$ exist $\Lambda \in SO^+(3,1)$, $\theta, \chi, \alpha \in \mathbb{R}$ such that

$M = \Lambda^{-1} \exp(\theta J_3 + \chi K_3) \Lambda$ or $M = \Lambda^{-1} \exp(\alpha(J_1 + K_2)) \Lambda$

In conclusion, for any $M \in SO^+(3,1)$ exist $\Lambda \in SO^+(3,1) \omega = (\omega_{\alpha\beta})_{\alpha,\beta} \in M_{4\times 4}(\mathbb{R})$ with $\omega = -\omega^T$, $M = \Lambda^{-1} \exp(\omega_{\alpha\beta} J^{\alpha\beta}) \Lambda$

Above we have already proven that

$\Lambda^{-1} J^{\alpha\beta} \Lambda = \Lambda^\alpha_{\ \mu} \Lambda^\beta_{\ \nu} J^{\mu\nu}$ and so, taking $\overline{\omega} = \Lambda^T \omega \Lambda$ we obtain:

$\overline{\omega}^T = -\overline{\omega}$, $M = \exp(\overline{\omega}_{\mu\nu} J^{\mu\nu})$

For any $M \in SO^+(3,1)$ exists $\omega \in M_{4\times 4}(\mathbb{R})$ such that $\omega = -\omega^T$ and $M = \exp(\omega_{\alpha\beta} J^{\alpha\beta})$

Let $\omega = (\omega_{\alpha\beta})_{\alpha,\beta} \in M_{4\times 4}(\mathbb{R})$ with $\omega = -\omega^T$ and we suppose $\det \omega \neq 0$. (11)

For $\Lambda \in SO^+(3,1)$ we have $\Lambda^{-1} \eta \omega \Lambda = \eta \Lambda^T \omega \Lambda$, $\Lambda^T \omega \Lambda = \eta \Lambda^{-1} \eta \omega \Lambda$. (12)

If $x, y \in M_{4\times 1}(\mathbb{C})$ and $x, y \neq 0$, $\mu, \lambda \in \mathbb{C}$ such that:

$\eta \omega x = \lambda x$, $\eta \omega y = \mu y$ then, because of (11) we have:

$\lambda,\mu\neq 0$ and $\bar{y}^T\omega x=\lambda\bar{y}^T\eta x$, $-\bar{y}^T\omega x=\bar{\mu}\bar{y}^T\eta x$. (the overline means that we are taking the complex conjugate) Therefore, if $\bar{y}^T\eta x\neq 0$ we must have $\lambda=-\bar{\mu}$.

Since we have also $\eta\omega\bar{x}=\bar{\lambda}\bar{x}$ it follows that if $\bar{x}^T\eta x\neq 0$ then $\lambda=-\bar{\lambda}$ (13)

and if $x^T\eta x\neq 0$ then $\lambda=-\lambda$.

Because we assumed $\det\omega\neq 0$ we must have $x^T\eta x=0$

for any $x\neq 0$ with $x\in M_{4\times 1}(\mathbb{C})$, $\lambda\in\mathbb{C}$, $\eta\omega x=\lambda x$

Let $x\in M_{4\times 1}(\mathbb{C})$, $x\neq 0$, $\lambda\in\mathbb{C}$ with $\eta\omega x=\lambda x$ and consider the case $\bar{x}^T\eta x=0$

Since $x^T\eta x=0$, for $u=\Re x$, $v=\Im x$ it follows $u^T\eta u=v^T\eta v=0$, $u^T\eta v=0$ and so: $u=cv$ or $v=c'u$, $c,c'\in\mathbb{R}$ and we can consider $x\in M_{4\times 1}(\mathbb{R})$, $\lambda\in\mathbb{R}^*$

We have $\det(\omega-\lambda\eta)=\det(\omega^T-\lambda\eta)=\det(\omega+\lambda\eta)$ and therefore

we can take $y\in M_{4\times 1}(\mathbb{R})$ with $y\neq 0$, $\eta\omega y=-\lambda y$

Supposing $y^T\eta y\neq 0$ it follows $\lambda=-\lambda=0$ which cannot be since we assumed $\det(\eta\omega)\neq 0$

Hence, in the considered case we have :

$x,y\in M_{4\times 1}(\mathbb{R})$ linear independent each of other with

$\lambda\in\mathbb{R}^*$, $y^T\eta y=x^T\eta x=0$, $\eta\omega x=\lambda x$, $\eta\omega y=-\lambda y$.

Taking $u=x+y$, $v=x-y$ we obtain $u^T\eta v=0$, $u^T\eta u=-v^T\eta v$.

Since *x* and *y* are independent, *u* and *v* are independent too and so we cannot have $u^T\eta u=-v^T\eta v=0$.

Hence we can take $u,v\in M_{4\times 1}(\mathbb{R})$ with $u^T\eta v=0$, $u^T\eta u=1$, $v^T\eta v=-1$ and $\eta\omega u=\lambda v$, $\eta\omega v=\lambda u$.

$\eta\omega$ invariates $\mathrm{Sp}(u,v)$.

If $\eta\omega$ invariates the subspace $V\subset M_{4\times 1}(\mathbb{R})$,for any $z\in V^\perp$ we have

$z^T\eta w=0$ for any $w\in V$ and so $(\eta\omega z)^T\eta w=-z^T\omega w=-z^T\eta w'=0$ for some $w'\in V$

Since $\det(\eta\omega)\neq 0$ weobtain that $\eta\omega$ invariates also $V^\perp$.

So $\eta\omega$ invariates $\mathrm{Sp}(u,v)^\perp$.

In the case $\bar{x}^T\eta x\neq 0$ we have $\lambda=i\mu$, $\mu\in\mathbb{R}^*$ and we take $u=\Re x$, $v=\Im x$.

We obtain: $\eta\omega u=-\mu v$, $\eta\omega v=\mu u$, $u^T\eta u=v^T\eta v\neq 0$, $u^T\eta v=0$

where we can take $u^T\eta u=v^T\eta v=-1$, u , v being independent since $\mu\neq 0$.

Therefore we have two Minkowski-orthogonal subspaces , in both considered cases, $\mathrm{Sp}(u_1,v_1)$ and $\mathrm{Sp}(u_2,v_2)$ invariated by $\eta\omega$ with $u_i^T\eta v_i=0$, $i=1,2$ one and only one of them having a vector, say v_1 with $v_1^T\eta v_1=1$ the other u_i,v_i having the Minkowski norm equal to -1.

So we have :

$$\eta\omega u_1=\lambda v_1 ,\ \eta\omega v_1=\lambda u_1 ,\ \eta\omega u_2=-\mu v_2 ,\ \eta\omega v_2=\mu u_2 ,\ \lambda,\mu\in\mathbb{R}^*$$

$$u_i^T\eta v_j=0 \text{ for } i,j=1,2;\ u_i^T\eta u_j=0 ,\ v_i^T\eta v_j=0 \text{ for } i\neq j ,\ i,j=1,2$$

$u_2^T\eta u_2=v_2^T\eta v_2=u_1^T\eta u_1=-1$, $v_1^T\eta v_1=1$ and we can choose u_i,v_i such that $v_{10}>0$.

Then we can take $\Lambda\in SO^+(3,1)$ with:

$\Lambda E_1=\varepsilon u_2$, $\Lambda E_2=\varepsilon v_2$, $\Lambda E_3=\varepsilon u_1$, $\Lambda E_0=v_1$, $\varepsilon\in\{\pm 1\}$.

For $\varphi=\Lambda^{-1}\eta\omega\Lambda$ we will have:

$\varphi E_1=-\mu E_2$, $\varphi E_2=\mu E_1$, $\varphi E_3=\varepsilon\lambda E_0$, $\varphi E_0=\varepsilon\lambda E_3$ and in the basis (E_1,E_2,E_3,E_0),

we have the matrix form :

$$\Lambda^T \omega \Lambda = \eta\varphi = \begin{pmatrix} 0 & -\mu & 0 & 0 \\ \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon\lambda \\ 0 & 0 & \varepsilon\lambda & 0 \end{pmatrix}$$

$\exp(\omega_{\alpha\beta} J^{\alpha\beta}) = \Lambda \exp(\omega_{\alpha\beta}\Lambda^{-1} J^{\alpha\beta}\Lambda)\Lambda^{-1} = \Lambda \exp(\omega_{\alpha\beta}\Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta} J^{\gamma\delta})\Lambda^{-1} =$
$= \Lambda \exp(2\mu J_3 - 2\varepsilon\lambda K_3)\Lambda^{-1}$
Since J_3 commutes with K_3 we will have $\exp(\omega_{\alpha\beta} J^{\alpha\beta}) \in SO^+(3,1)$,
if as we assumed $\omega = -\omega^T \in M_{4\times 4}(\mathbb{R})$ with $\det\omega \neq 0$.
If we have $\overline{\omega} = -\overline{\omega}^T \in M_{4\times 4}(\mathbb{R})$ and $\det\overline{\omega} = 0$ we observe that the set
$A = \{\omega \in M_{4\times 4}(\mathbb{R}) | \omega = -\omega^T \text{ , } \det\omega \neq 0\}$ is dense in $\{\omega \in M_{4\times 4}(\mathbb{R}) | \omega = -\omega^T\} = \overline{A}$.
The function $M_{4\times 4}(\mathbb{R}) \ni \omega \to \exp(\omega_{\alpha\beta} J^{\alpha\beta}) \in M_{4\times 4}(\mathbb{R})$ being continuous,
since $SO^+(3,1)$ is closed in $M_{4\times 4}(\mathbb{R})$ it follows $\exp(\overline{\omega}_{\alpha\beta} J^{\alpha\beta}) \in SO^+(3,1)$ for any $\overline{\omega} \in \overline{A}$

The above proven results lead to the following three facts:
i) We have 6 independent matrices $\{H_k\}_{k=\overline{1,6}}$ where
$H_k = -\frac{1}{2}\epsilon_{ijk} J^{ij} = J_k$, $H_{k+3} = -J^{0k}$ for $k = \overline{1,3}$
ii) We have a surjective C^∞ class function
$\Phi : \mathbb{R}^6 \to SO^+(3,1)$, $\Phi((\psi_s)_s) = \exp(\psi_s H_s)$ such that $\operatorname{rank}\left(\frac{\partial \Phi_{pq}}{\partial \psi_k}\right)_{pq,k} = 6$
with $p,q = \overline{0,3}$, $k = \overline{1,6}$
iii) Φ is local injective (Since $\det\left(\frac{\partial e^W_{pq}}{\partial t_{ij}}\right)_{pq,ij} \neq 0$, $p,q,i,j = \overline{0,3}$
for any $W \in M_{4\times 4}(\mathbb{R})$, $W = (t_{ij})_{i,j}$)

As we proved for the rotation group *SO(n)* we conclude that the manifold structure on $SO^+(3,1)$(with the topology induced from $M_{4\times 4}(\mathbb{R})$) is equivalent to a structure given by the mappings $((\psi_s)_s) \to \exp(\psi_s H_s)$.
Having the continuous surjective function Φ we find $SO^+(3,1)$ as a 6-dimensional connected Lie group (as well as by the mapping $(\vec{\theta}, \vec{\chi}) \to \exp(\vec{\theta}\vec{J})\exp(\vec{\chi}\vec{K})$).

For $U \in M_{n\times n}(\mathbb{C})$, $n \in \mathbb{N}^*$ we denote U^+ the conjugate transpose of U .
Let $SU(n) = \{U \in M_{4\times 4}(\mathbb{C}) | U^+ U = \mathbf{I} \text{ , } \det U = 1\}$
Consider $M_{n\times n}(\mathbb{C})$ as its natural complex Hilbert space .
Then if $x \in M_{(n+1)\times(n+1)}(\mathbb{C})$, $\lambda \in \mathbb{C}$, $U \in SU(n+1)$ with $Ux = \lambda x$ we will have that U invariates $\operatorname{Sp}(x)$ and also $\operatorname{Sp}(x)^{\perp}$ and $\lambda\overline{\lambda} = 1$.
Therefore we can obviously prove by induction (in a sampler way as we did for *SO(n)*) that for any $U \in SU(n)$ exists $H \in M_{n\times n}(\mathbb{C})$ with $H^+ = H$, $U = \exp(iH)$. (14)
Since $\operatorname{tr}(H) = \operatorname{tr}(JHJ^{-1})$, $\det U =$ for any $J \in M_{n\times n}(\mathbb{C})$ with $\det J \neq 0$, taking *H* in the normal Jordan form, from (14) we deduce for H that $\operatorname{tr} H = 0$.
From the way we proved it , it is obvious that the relation (3’) works even for complex *W*.
Therefore are no difficulties in proving that we have a surjective and local injective mapping
$\Psi : \mathbb{R}^r \to SU(n)$, $\Psi(\varphi) = \exp(i\varphi_a T_a)$ where $\varphi = (\varphi_a)_{a=\overline{1,r}}$, $r = n^2 - 1$,

$(T_a)_a$ is a basis of the real vector space $S=\{H\in M_{4\times 4}(\mathbb{C}) | H^+=H \text{ , tr} H=0\}$.

For $n=2$ we can take $(T_a)_a=(\sigma_i)_{i=\overline{1,3}}$ the Pauli matrices (see Chap. Representations of the rotations group and the restricted Lorentz group. Spin representations).

Let $M=\exp(\omega_{\mu\nu}J^{\mu\nu})$ with $\omega=-\omega^T\in M_{4\times 4}(\mathbb{R})$

Then,as we proved above, if $\det\omega\neq 0$ we can find $\Lambda\in SO^+(3,1)$ such that

$$\Lambda^T\omega\Lambda=\omega' \text{ , } \omega'_{\alpha\beta}=0 \text{ for } (\alpha,\beta)\notin\{(1,2),(2,1),(0,3),(3,0)\} \text{ and } \quad (15)$$

$$\Lambda^{-1}M\Lambda=\exp(-2\omega'_{12}J_3-2\omega'_{03}K_3)$$

We have the representation

$S:SO^+(3,1)\to M_{4\times 4}(\mathbb{C})$ such that for any $M\in SO^+(3,1)$, $S=S(M)$ satisfies

$S^{-1}\gamma^\mu S=M_{\mu\nu}\gamma^\nu$ for $\mu=\overline{0,3}$ (16) (see Chap. Representations of the rotations group and the restricted Lorentz group. Spin representations).

$$S(\exp(\theta J_3))=\cos(\frac{\theta}{2})\mathbf{I}+\sin(\frac{\theta}{2})\gamma^1\gamma^2=\exp(\frac{\theta}{4}[\gamma^1,\gamma^2])$$

$$S(\exp(\chi K_3))=\cosh(\frac{\chi}{2})\mathbf{I}+\sinh(\frac{\chi}{2})\gamma^0\gamma^3=\exp(\frac{\chi}{4}[\gamma^0,\gamma^3]) \quad (17)$$

where $[A,B]=AB-BA$ denotes the commutator of A and B

We denote $\sigma^{\mu\nu}=\frac{i}{2}[\gamma^\mu,\gamma^\nu]$.

Since $[J_3,K_3]=0$ and $[\sigma^{12},\sigma^{03}]=0$, from (15) and (17) , after some calculus we obtain:

$$S(M)=\exp(\frac{i}{2}2\omega'_{12}S(\Lambda)\sigma^{12}S(\Lambda)^{-1}+\frac{i}{2}2\omega'_{03}S(\Lambda)\sigma^{03}S(\Lambda)^{-1}) \text{ . } (18)$$

From (16) we can deduce $S(\Lambda)^{-1}\sigma^{\mu\nu}S(\Lambda)=\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\sigma^{\alpha\beta}$ and so (15) and (18) will lead to:

$S(M)=\exp(\frac{i}{2}\omega_{\alpha\beta}\sigma^{\alpha\beta})$ if as we assumed $\det\omega\neq 0$

If $\det\omega=0$ we have $\omega=\lim_{n\to\infty}\omega_n$ with $\omega_n=-\omega_n^T\in M_{4\times 4}(\mathbb{R})$, $\det\omega_n\neq 0$ and

since the representation S is continuous, for $M_n=\exp(\omega_{n\alpha\beta}J^{\alpha\beta})$ we have

$\lim_{n\to\infty}S(M_n)=S(M)$ deducing $S(M)=\exp(\frac{i}{2}\omega_{\alpha\beta}\sigma^{\alpha\beta})$ for any $\omega=-\omega^T\in M_{4\times 4}(\mathbb{R})$.

The Dirac spinorial function $\psi=(\psi_\alpha)_\alpha(x)$ (as a column 4x1 matrix) ,

$x=(x^\alpha)_\alpha$ space-time coordinates, which satisfies the Dirac equation

$$i\gamma^\mu\partial_\mu\psi-m\psi=0$$

transforms under a Lorentz coordinates transformation

$x'^\mu=M_{\mu\nu}x^\nu$ according to $\psi'=S(M)\psi$ and considering

$M=\exp(\omega_{\alpha\beta}J^{\alpha\beta})$, $\overline{\psi}=\psi^+\gamma^0$ with ψ^+ the complex conjugate transpose of ψ

we have for the transformation of the conserved current , $J^\mu=\overline{\psi}\gamma^\mu\psi$, the expression:

$$J'^\mu=\psi^+S^+(M)\gamma^0\gamma^\mu S(M)\psi=\psi^+\exp(-\frac{i}{2}\omega_{\alpha\beta}\sigma^{+\alpha\beta})\gamma^0\gamma^\mu\exp(\frac{i}{2}\omega_{\alpha\beta}\sigma^{\alpha\beta})\psi$$

We have $\sigma^{+\alpha\beta}=\gamma^0\sigma^{\alpha\beta}\gamma^0$ and so we obtain:

$$J'^\mu=\psi^+\gamma^0\exp(-\frac{i}{2}\omega_{\alpha\beta}\sigma^{\alpha\beta})\gamma^\mu\exp(\frac{i}{2}\omega_{\alpha\beta}\sigma^{\alpha\beta})\psi=\psi^+\gamma^0S(M)^{-1}\gamma^\mu S(M)\psi=M_{\mu\nu}\overline{\psi}\gamma^\nu\psi \text{ .}$$

Therefore, the conserved current transforms like a contravariant Lorentz vector.