Feynman amplitudes and lattice gauge theory

We consider, (by suitable choosing of length, time and charge units) that the reduced Planck constant and the speed of light in vacuum constant are equal to 1.

$\hbar=1$, $c=1$

For a quantum field system described by field operator functions

$\hat{\varphi}=\hat{\varphi}(t,\vec{x})$ $\quad(x=(t,\vec{x})=(x_{\alpha})_{\alpha=\overline{0,3}}$ space-time coordinates $)$ in the Minkowski space with signature (+ , - , - , -) , $\eta=(\eta^{\alpha\beta})_{\alpha,\beta=\overline{0,3}}$ Minkowski metric coefficients and a Lagrangian density

$\mathscr{L}=\mathscr{L}(\varphi,\partial\varphi)$ with $\varphi=(\varphi_i)_i$ the action is

$S(\varphi)=\int\mathscr{L}(\varphi,\partial\varphi)d^4x$ and we can have $S(\varphi)=\int\left(\frac{1}{2}\varphi_i M_{ij}\varphi_j-V(\varphi)\right)d^4x$

where M_{ij} is a differential operator and we use Einstein summation convention.

We can make a discretization of the quantum field in the form

$q(t)=(q^k(t))_{k=\overline{1,M}}=(\varphi(t,an_1,an_2,an_3))_{n1,n2,n3\in\mathbb{Z}}$

The momentum field operator function is

$\hat{\pi}=\frac{\partial\widehat{\mathscr{L}}}{\partial(\partial_0\varphi)}$ and corresponds in discretization to the momentum coordinates

$p(t)=(\pi(t,an_1,an_2,an_3))_{n1,n2,n3\in\mathbb{Z}}$ of a discretized phase space evolution $(p(t),q(t))$ with a Hamiltonian operator given by the discretized correspondent of the expression

$\hat{H}(t)=\int(\hat{\pi}(t,\vec{x})\partial_0\hat{\varphi}(t,\vec{x})-\mathscr{L}(\hat{\varphi},\partial\hat{\varphi})(t,\vec{x}))d^3\vec{x}$ which we denote

$\hat{H}=\hat{H}(\hat{p},\hat{q})$.

As we know, (see Chap. Quantum statistical ensemble) for any observable $A=A(t)$ for the expectation value $\langle A\rangle_t=\mathrm{tr}(\rho A)$ $(\rho$ the density operator $)$ we have an evolution equation

$\frac{d}{dt}\langle A\rangle_t=i\langle[\hat{H},A]\rangle_t+\langle\partial_0 A\rangle_t$ $\quad([\hat{H},A]=\hat{H}A-A\hat{H}$ the commutator $)$

Since $\hat{p}$, $\hat{q}$ not depend explicitly on time we can consider evolution equations for $\hat{p}$, $\hat{q}$ observables functions $A(\hat{p})$, $A(\hat{q})$ like

$\frac{d}{dt}A(\hat{p})(t)=i[\hat{H},A(\hat{p})](t)$; $\frac{d}{dt}A(\hat{q})(t)=i[\hat{H},A(\hat{q})](t)$

and so $A(\hat{p})$, $A(\hat{q})$ evolve like

$$A(\hat{q})(t)=\exp(i\hat{H}t)A(\hat{q})(0)\exp(-i\hat{H}t)\ ;\quad A(\hat{p})(t)=\exp(i\hat{\hat{H}}t)A(\hat{p})(0)\exp(-i\hat{H}t) \qquad (1)$$

Therefore in the continuum limit of the discretization $(a\to 0)$ we have an evolution of operators :

$A(\hat{\varphi})(t)=\exp(i\hat{H}t)A(\hat{\varphi})(0)\exp(-i\hat{H}t)$

Given the final and initial states $\varphi_F=\varphi_F(\vec{x})$, $\varphi_I=\varphi_I(\vec{x})$ corresponding in the discretization to q_F respective q_I we have the transition amplitude for the system from state φ_I at $t=0$ to state φ_F at $t=T$: $\langle \varphi_F|\exp(-i\hat{H}t)|\varphi_I\rangle = A$ (do not confuse with the observable A)

$|A|^2$ is the probability for the system to be in state φ_F at time $t=T$ if at time $t=0$ it was measured in state φ_I , since from state φ_I the system evolves according to Schroedinger equation like

$\exp(-i\hat{H}t)\left|\varphi_I\right\rangle$

Considering a normalization of *p* and *q* states in which

$\langle q'|q\rangle = \delta(q'-q)$, $\langle q|p\rangle = \exp(ipq)$ and

$\int |q\rangle\langle q| d^M q = \boldsymbol{I}$ and $\int |p\rangle\langle p| \frac{d^M p}{(2\pi)^M} = \boldsymbol{I}$

and taking $\delta t = \frac{T}{N}$ we have for $t_1 \in [l\,\delta t, (l+1)\delta t]$, if we consider that

$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$ the following relations :

$$\langle q_F|\exp(-i\hat{H}t)A(\hat{q})(t_1)|q_I\rangle = \left(\prod_{j=1}^{N-1}\int dq^j\right)\langle q_F|\exp(-i\hat{H}\delta t)|q_{N-1}\rangle\left\langle q_{N-1}\right|$$

$$\exp(-\hat{H}\delta t|q_{N-2})\rangle \ldots \langle q_{l+1}|\exp(-i\hat{H}\delta t)A(\hat{q})(0)|q_l\rangle \ldots \langle q_1|\exp(-i\hat{H}\delta t)|q_I\rangle$$

$$\langle q_{l+1}|\exp(-i((\hat{p}^2/2m)+V(\hat{q}))\delta t)|A(\hat{q})(0)|q_l\rangle =$$

$$\frac{\int d^M p}{(2\pi)^M}\exp(-i\delta t((p^2/2m)+V(q_l)))A(q_l)\langle q_{l+1}|p\rangle\langle p|q_l\rangle =$$

$$\left(\frac{-im}{2\pi\delta t}\right)^{M/2} A(q_l)\exp((im(q_{l+1}-q_l)^2/2\delta t) - iV(q_l)\delta t) \text{ and so}$$

$$\langle q_F|\exp(-i\hat{H}T)A(\hat{q})(t_1)|q_I\rangle =$$

$$\left(\frac{-im}{2\pi\delta t}\right)^{MN/2}\left(\prod_{k=1}^{N-1} dq_k\right)\exp(i\delta t(\sum_{j=0}^{N}(m/2)((q_{j+1}-q_j)/\delta t)^2) - V(q_j))A(q_l) = \tag{2}$$

$$C\int Dq(t)\exp(i\int_0^T((1/2)m\dot{q}^2 - V(q))dt)A(q(t_1)) =$$

$$C\int Dq(t)\exp(iS(q))A(q(t_1))$$

where $Dq(t)$ stands for integration over all paths $q=q(t)$ with $q(0)=q_I$, $q(T)=q_F$

(In deriving (2) we used the Fresnel integrals :

$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$)

Therefore in the same way, for

$A_i = A_i(\hat{\varphi})$, $i=\overline{1,n}$ operatorial functions, we will have :

$$\langle \varphi_F|\exp(-i\hat{H}T)\mathcal{T}(A_1(t_1,\vec{x}_1)\ldots A_n(t_n,\vec{x}_n))|\varphi_I\rangle = \\ C\int D\varphi\exp(iS(\varphi))\mathcal{T}(A_1(\varphi(t_1,\vec{x}_1))\ldots A_n(\varphi(t_n,\vec{x}_n))) \tag{3}$$

where *C* is a (discretization dependent) constant and $D\varphi$ stands for integration over all paths $\varphi=\varphi(t)$ with $\varphi(0)=\varphi_I$ and $\varphi(T)=\varphi_F$, $\varphi(t)=\varphi(t,.)$ and if $(q^i)_{i=\overline{1,M}}=(\varphi(an_0,an_1,an_2,an_3))_{n_0,n_1,n_2,n_3\in\mathbb{Z}}$ is a discretization of the field, we define :

$\int D\varphi ... = \int \prod_{i=1}^{M} dq^i ...$ and also:

$$T\,(A_1(\hat{\varphi}(t_1,\vec{x}_1))...A_n(\hat{\varphi}(t_n,\vec{x}_n)))=$$
$$\sum_{\sigma\in Sn}\left(\prod_{j=1}^{n-1}\theta(t_{\sigma(j)}-t_{\sigma(j+1)})\right)A_1(\hat{\varphi}(t_{\sigma(1)},\vec{x}_{\sigma(1)}))...A_n(\hat{\varphi}(t_{\sigma(n)},\vec{x}_{\sigma(n)}))$$

with θ the Heaviside function $\theta(t)=\begin{cases}1 & \text{for } t>0\\ 0 & \text{for } t<0\end{cases}$

In the formula above, we take the 0 argument value of the Heaviside function to be 1 and divide the right side of the identity for every case of k times occurrence of the same value of t_j by k! .

The Euler – Lagrange equations

$d_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}-\frac{\partial\mathcal{L}}{\partial\varphi}=0$ and the commutation rules $[\hat{p}^k,\hat{q}^j]=i\,\delta_{jk}$

which commutation rules translated to the continuum limit become

$[\hat{\pi}^k(t,\vec{x}),\hat{\varphi}^j(t,\vec{x})]=\delta^3(\vec{x})\delta_{kj}$ with $\hat{\pi}^k=\widehat{\frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}}$ lead to:

$$\text{a) } \hat{\varphi}(t,\vec{x})=\frac{1}{(2\pi)^{3/2}}\int\frac{1}{(2\,\omega_k)^{1/2}}\Big(a(\vec{k})\exp(-i(\omega_k t-\vec{k}\,\vec{x}))+ b^+(\vec{k})\exp(i(\omega_k t-\vec{k}\,\vec{x}))\Big)d^3\vec{k} \qquad (4a)$$

for a complex boson free field theory with :

$\mathcal{L}(\varphi,\partial\varphi)=(\partial\varphi^+)(\partial\varphi)-m^2\varphi^+\varphi$, $\omega_k=\sqrt{\vec{k}^2+m^2}$; a,a^+ and b,b^+ annihilation and creation operators for the particle respective the antiparticle of the field satisfying commutation relations :

$[a(\vec{k}),a^+(\vec{k}')]=[b(\vec{k}),b^+(\vec{k}')]=\delta^3(\vec{k}-\vec{k}')$, $[a(\vec{k}),b(\vec{k}')]=0$

$$\text{b) } \widehat{A}_\mu(t,\vec{x})=\sum_s\frac{1}{(2\pi)^{3/2}}\int\frac{1}{(2\omega_k)^{1/2}}\Big(\varepsilon_\mu(\vec{k},s)a(\vec{k},s)\exp(-i(\omega_k t-\vec{k}\,\vec{x}))+ \varepsilon_\mu(\vec{k},s)a^+(\vec{k},s)\exp(i(\omega_k t-\vec{k}\,\vec{x}))\Big)d^3\vec{k} \qquad (4b)$$

where $(\varepsilon_\mu(\vec{k},s))_\mu$ are the polarization vectors $s=\overline{1,3}$ for $m\neq0$ and $s=1,2$ for *m* = 0 . Also we have $[a(\vec{k},s),a^+(\vec{k}',s')]=\delta^3(\vec{k}-\vec{k}')\delta_{ss'}$

For $m\neq0$ in the rest frame $k=(m,0,0,0)$ we have $\varepsilon(\vec{0},s)=(0,(\delta_{is})_i)$

By Lorentz invariance it follows that :

$k^\mu\varepsilon_\mu=0$, $\varepsilon^\mu(\vec{k},s)\varepsilon_\mu(\vec{k},s')=-\delta_{ss'}$ and $\sum_s\varepsilon_\mu(\vec{k},s)\varepsilon_\lambda(\vec{k},s)=K_{\mu\lambda}$ with (4)

$$K_{\mu\lambda}=-\eta_{\mu\lambda}+(k_\mu k_\lambda/m^2) \text{ if } m\neq 0$$

If $m=0$, for $K_{\mu\lambda}$ to be well determined by k we give to the massless bosons a hypothetical mass and compute with tending to 0 mass value.

The b) case is the case of a gauge vector boson free field theory

The gauge bosons free field Lagrangian density is (see Chap. Non-abelian gauge theory) given by :

$$\mathscr{L}((A^a_\mu,\partial A^a_\mu)_{a,\mu})=-\frac{1}{4}(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu)(\partial^\mu A^{a\nu}-\partial^\nu A^{a\mu})-$$
$$-\frac{1}{2}g(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu)f^{abc}A^{b\mu}A^{c\nu}-$$
$$-\frac{1}{4}g^2 f^{abc}f^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu}+\frac{1}{2}M_a^2 A^a_\mu A^{a\mu}$$

M_a are the gauge bosons masses and f^{abc} are the structure coefficients of the gauge group Lie algebra , having normalized generators

$(T^a)_a$ with $[T^b,T^c]=if^{abc}T^a$, $\mathrm{tr}(T^aT^b)=\frac{1}{2}\delta_{ab}$ and T^a hermitian traceless.

c) For spin ½ fermions in a free field theory the Lagrangian density is the Dirac Lagrangian density :

$\mathscr{L}(\psi,\partial\psi)=\overline{\psi}(i\gamma^\mu\partial_\mu-m)\psi$ with γ^μ the gamma matrices, $\overline{\psi}=\psi^+\gamma^0$

$\psi=(\psi_\alpha)_{\alpha=\overline{0,3}}(t,\vec{x})$ Dirac spinor field.

$$\widehat{\psi_\alpha}(t,\vec{x})=\frac{1}{(2\pi)^{3/2}}\int\frac{1}{(E_p/m)^{1/2}}\left(\sum_s u_\alpha(p,s)b(p,s)\exp(-ipx)+ v_\alpha(p,s)d^+(p,s)\exp(ipx)\right)d^3\vec{p} \quad (4c)$$

where $s=1,2$ and $p=(p_\mu)_{\mu=\overline{0,3}}=(p_0,\vec{p})$, $px=p_\mu x^\mu$, $E_p=p_0=\sqrt{\vec{p}^2+m^2}$

$(i\gamma^\mu\partial_\mu-m)\psi=0$ and $(\not{p}-m)u(p,s)=0$, $(\not{p}+m)v(p,s)=0$ (4')

since by Euler – Lagrange equations , the spinor field satisfies the Dirac equations with $\not{p}=\gamma^\mu p_\mu$

The annihilation and creation operators for particles respective antiparticles b,b^+ respective d,d^+ satisfy anti-commutation relations :

$\{b(p,s),b^+(p',s')\}=\{d(p,s),d^+(p',s')\}=\delta_{ss'}\delta^3(\vec{p}-\vec{p}')$ (4'')

$\{b(p,s),d(p',s')\}=\{b(p,s),b(p',s')\}=\{d(p,s),d(p',s')\}=0$ (4'')

with $\{A,B\}=AB+BA$, the anti-commutator

The normalized *u* and *v* functions are so that in the rest frame

$p=(m,0,0,0),u(p,1)=(1,0,0,0),u(p,2)=(0,1,0,0),v(p,1)=(0,0,1,0)$

$v(p,2)=(0,0,0,1)$ as column vectors and by Lorentz invariance we will have:

$\overline{u}(p,s)u(p,s')=\delta_{ss'}$, $\overline{v}(p,s)v(p,s')=-\delta_{ss'}$

$\overline{u}(p,s)v(p,s')=\overline{v}(p,s)u(p,s')=0$

and $\sum_s u_\alpha(p,s)\overline{u_\beta}(p,s)=\left(\frac{\not p+m}{2m}\right)_{\alpha\beta}$ (4''')

and $\sum_s v_\alpha(p,s)\overline{v_\beta}(p,s)=\left(\frac{\not p-m}{2m}\right)_{\alpha\beta}$ (4''')

Considering a perturbed by sources $J=J(t,\vec{x})$ Lagrangian density $\mathscr{L}=\mathscr{L}(\varphi,\partial\varphi)+J\varphi$, in the discretization we may have $S(\varphi)=\int\left(\frac{1}{2}\varphi_i M_{ij}\varphi_j+J_i\varphi_i\right)d^4x=\frac{1}{2}q^T A q+J^T q$ with A a symmetric real non-singular matrix.

$\frac{1}{2}q^T A q+J^T q=\frac{1}{2}(q^T+J^T A^{-1})A(q+A^{-1}J)-\frac{1}{2}J^T A^{-1}J$ (5) and so

$Z(J)=\int D\varphi\exp(iS(\varphi))=C(\int d^M q\exp((i/2)q^T Aq))\exp(-(i/2)J^T A^{-1}J)$.

Diagonalizing A and considering the already mentioned Fresnel integrals we obtain

$\int d^M q\exp((i/2)q^T Aq)=\left(\frac{(2\pi i)^M}{\det A}\right)^{1/2}$

A^{-1} corresponds to the propagator $D=D(x-y)$ which in the continuum limit satisfies $M_{ij}D_{jk}(x)=\delta_{ik}\delta^4(x)$ (6)

and we have $Z(J)=Z(J=0)\exp((-i/2)\int J_j(x)D_{jk}(x-y)J_k(y)d^4x\,d^4y)$

The ground state corresponds to the state with lowest energy possible , no perturbations in the field (only vacuum fluctuations) : $\varphi\equiv 0$ and we denote it $|0\rangle$. Taking $\varphi_F=\varphi_I=|0\rangle$, according to (3) we will have:

$$\langle 0|\exp(-i\hat{H}T)T(\varphi(x_1)\ldots\varphi(x_n))|0\rangle=$$

$$C\int D\varphi\exp(i\int\mathscr{L}(\varphi,\partial\varphi)d^4x)\varphi(x_1)\ldots\varphi(x_n)=$$

$$\left.\left(\frac{\delta^n}{\delta iJ(x_1)\ldots\delta iJ(x_n)}C\int D\varphi\exp(i\int\mathscr{L}(\varphi,\partial\varphi)+J\varphi d^4x)\right)\right|_{J=0}=Z(J=0)$$

$$\left.\left(\frac{\delta^n}{\delta iJ(x_1)\ldots\delta iJ(x_n)}\exp((-i/2)\int J_k(x)D_{kl}(x-y)J_l(y)d^4x\,d^4y)\right)\right|_{J=0}$$

where $\frac{\delta}{\delta iJ(x_k)}$ must be understood as a partial derivative with respect to

$iJ(x_k)d^4x$.

In the case of fermion fields Lagrangian density $\mathscr{L}=\mathscr{L}(\psi,\partial\psi)$, because the spinor fields are complex we have ψ and $\overline{\psi}$ as independent integration variables and a perturbed Lagrangian density form by spinor sources η and $\overline{\eta}$ as below:

$\mathscr{L}(\psi,\partial\psi)+\overline{\psi}\eta+\overline{\eta}\psi$ and the path integral:

$$Z(\eta,\overline{\eta})=\int D\psi D\overline{\psi}\exp\left(i\int\left(\mathscr{L}(\psi,\partial\psi)+\overline{\psi}\eta+\overline{\eta}\psi\right)d^4x\right) \quad (7)$$

We have: $\mathscr{L}(\psi,\partial\psi)=\overline{\psi}K\psi$, $K=i\partial\!\!\!/-m$

$$\mathscr{L}(\psi,\partial\psi)+\overline{\eta}\psi+\overline{\psi}\eta=(\overline{\psi}+\overline{\eta}K^{-1})K(\psi+K^{-1}\eta)-\overline{\eta}K^{-1}\eta \quad (8)$$

The propagator $S=(i\partial\!\!\!/-m)^{-1}$, $S=(S_{\alpha\beta}(x))_{\alpha,\beta=\overline{0,3}}$

$S = D^{\text{fer}}$,fermion propagator,

satisfies $(i\partial\!\!\!/-m)S(x)=\delta^4(x)$

and we have therefore $S(x)=\int\frac{1}{(2\pi)^4}\exp(-ipx)\frac{p\!\!\!/+m}{p^2-m^2+i\varepsilon}d^4p$ (8’)

with $\varepsilon>0$, $\varepsilon\rightarrow0$. Using residues theorem in the integration above over p_0 integration variable we obtain :

$$iS(x)=\frac{1}{(2\pi)^3}\int(2E_p)^{-1}\Big(\theta(x^0)(p\!\!\!/+m)\exp(-ipx)-$$
$$-\theta(-x^0)(p\!\!\!/-m)\exp(ipx)\Big)d^3\vec{p} \quad (8'')$$

where in the above expression we take $E_p=p_0=\sqrt{\vec{p}^2+m^2}$

Considering the (4’’) anti-commutation relations we can take therefore the $(\overline{\psi}_\alpha(x))_{\alpha,x}$ and $(\psi_\alpha(x))_{\alpha,x}$ integration variables as two sets of independent Grassmann integration variables. Grassmann numbers are defined such that if η and ξ belong to the same set of Grassmann numbers then $\eta\xi=-\xi\eta$.

Therefore the most general function of a Grassmann number is $f=f(\eta)=a+b\eta$ with a , b ordinary numbers .

Since for η , ξ Grassmann variables we must have $\int d\eta f(\eta+\xi)=\int d\eta f(\eta)$ and so $\int d\eta b\xi=0$ for any ξ and we have $\int d\eta=0$ for η Grassmann integration variable.

Since given three Grassmann variables χ,η,ξ we have $\chi(\eta\xi)=(\eta\xi)\chi$ we conclude that the product of two Grassmann numbers must be an ordinary number and thus the integral $\int\eta d\eta$ is an ordinary number which is taken to be equal to a normalization constant.

Therefore, if $\eta=(\eta_1,\ldots,\eta_N)$ and $\overline{\eta}=(\overline{\eta}_1,\ldots,\overline{\eta}_N)$ are sets of independent Grassmann variables and $s=(s_1,\ldots s_N),r=(r_1,\ldots,r_N)$ not depend on η , $\overline{\eta}$, *A* is a N x N matrix then we can derive $\int d\eta d\overline{\eta}\exp((\overline{\eta}+s)A(\eta+r))=\int d\eta d\overline{\eta}\exp(\overline{\eta}A\eta)=C\det A$ with C a normalization constant. (8’’’)

Hence, considering (8) the relation (7) becomes

$$Z(\eta,\overline{\eta})=Z(\eta=0)\exp\left(-i\int\overline{\eta}_\alpha(x)S_{\alpha\beta}(x-y)\eta_\beta(y)d^4x\,d^4y\right)$$

and we have $Z(\eta=0)=C\det(i\partial\!\!\!/-m)$,

with $\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3$ we will have also:

$$\det(i\partial\!\!\!/-m)=\exp(\mathrm{tr}\ln(i\partial\!\!\!/-m))$$
$$\mathrm{tr}\ln(i\partial\!\!\!/-m)=\mathrm{tr}\ln(\gamma^5(i\partial\!\!\!/-m)\gamma^5)=\mathrm{tr}\ln(-i\partial\!\!\!/-m)=(1/2)4\,\mathrm{tr}\ln(\partial^2+m^2)$$

The factor of 4 appears because at the left member we have the trace of a 4x4 matrix.

Recall that $Z(\eta=0)=\int D\psi D\overline{\psi}\exp(i\int\mathscr{L}(\psi,\partial\psi)d^4x)=\langle 0|\exp(-i\hat{H}T)|0\rangle$ (9)

$($ with $T\to\infty$ understood so that we integrate over all of spacetime in (9) $)$

and so if $E=\langle 0|\hat{H}|0\rangle$ is the energy of the vacuum we will have:

$$iET=-2\,\mathrm{tr}\ln(\partial^2+m^2)+AVT=-2\int d^4x\langle x|\ln(\partial^2+m^2)|x\rangle+AVT=$$
$$-2\int d^4x\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\langle x|k\rangle\langle k|\ln(\partial^2+m^2)|q\rangle\langle q|x\rangle+AVT$$

Since in the momentum space normalization $\langle k|q\rangle=(2\pi)^4\delta^4(k-q)=VT$

(V space volume , T time interval of the considered field domain) we obtain

$i\frac{E}{V}=-2\int\frac{d^4k}{(2\pi)^4}\ln(k^2-m^2+i\varepsilon)+A'$ where A , A' are infinite constants corresponding to the multiplicative factor C (and changing the sign under the logarithm)

Let $\frac{A'}{2}=\int\frac{d^4k}{(2\pi)^4}\ln(k^2-m'^2+i\varepsilon)$ and we will have

$$\frac{E}{V}=2i\int\frac{d^3\vec{k}}{(2\pi)^3}\int\frac{d\omega}{2\pi}\ln\left(\frac{\omega^2-\omega_k^2+i\varepsilon}{\omega^2-\omega'^2_k+i\varepsilon}\right)$$

We treat the (convergent) integral over ω by integrating by parts and then by residues theorem, obtaining:

$$\frac{E}{V}=-2i\int\frac{d^3\vec{k}}{(2\pi)^3}\int\frac{d\omega}{2\pi}\left(\frac{2\omega_k^2}{\omega^2-\omega_k^2+i\varepsilon}-\frac{2\omega'^2_k}{\omega^2-\omega'^2_k+i\varepsilon}\right)=$$
$$=\int\frac{d^3\vec{k}}{(2\pi)^3}(-2(\omega_k-\omega'_k))$$

$($ where we defined $\omega_k=\sqrt{\vec{k}^2+m^2},\omega'^2_k=\sqrt{\vec{k}^2+m'^2})$

Restoring the Planck constant through dimensional analysis we have

$E_0=-\int\frac{d^3\vec{x}\,d^3\vec{p}}{h}\sum_s 2\left(\frac{1}{2}E_p\right)$ with $E_p=\sqrt{\vec{p}^2c^2+m^2c^4}$

The infinite additive term E_0 is precisely the analogue of the zero point energy of the quantum harmonic oscillator but for the Dirac field , as we see , comes with a peculiar minus sign. For each spin and for the electron and positron separately (hence the factor of 2) we have an energy (- 1 / 2) E_p in each unit-size phase-space cell $(1/h^3)\,d^3x\,d^3p$.

To compute a Feynman amplitude for all modulo time ordering equivalence classes of Feynman diagrams with

$(x_i^a)_{i=\overline{1,s}}$ outgoing legs end vertices and

$(x_i^b)_{i=\overline{1,n}}$ incoming legs end vertices and

$(x_i)_{i=\overline{1,m}}$ interaction vertices, $(p_i)_{i=\overline{1,s}}$ outgoing momenta and

$(q_i)_{i=\overline{1,n}}$ incoming momenta, considering that $\hat{H}|0\rangle = V_0|0\rangle$

with V_0 constant vacuum energy, which by measuring energy at this level can be considered to be equal to 0, we have to compute

$$\int P((x^a),(x),(x^b))(\prod_{j=1}^{s}\exp(i p_j x_j^a) d^4 x_j^a)(\prod_{k=1}^{n}\exp(-i q_k x_k^b) d^4 x_k^b)(\prod_{l=1}^{m} d^4 x_l)$$

where $P((x^a),(x),(x^b)) = \langle 0| T((\prod_{j=1}^{s}\hat{\varphi}(x_j^a))(\prod_{l}\hat{\varphi}(x_l))(\prod_{k=1}^{n}\hat{\varphi}(x_k^b)))|0\rangle$

and "amputate" the external legs ,(i . e. multiply with

$(\prod_{j=1}^{s}(p_j^2 - m^2 + i\varepsilon))(\prod_{k=1}^{n}(q_k^2 - m^2 + i\varepsilon))$, the incoming and respective outgoing particles being considered on mass shell).

Notice that the interaction vertices must not be all distinct and so we will integrate over the set of distinct x_l , $l=\overline{1,m}$ and $\prod_l \hat{\varphi}(x_l)$ represents the product of a exponential expansion coefficient and the taken interaction vertices terms from the expression of the interaction Lagrangian density

$\tilde{\mathscr{L}}(\varphi,\partial\varphi) = \mathscr{L}(\varphi,\partial\varphi) +$ interaction terms

As we derived above , for

$P((x^a),(x),(x^b))$ we have a Wick contraction computation from the expression

$$\tilde{C}\left(\frac{\delta^{n+s+m}}{\delta i J(x_1^a)\ldots\delta i J(x_l)\ldots\delta i J(x_n^b)}\exp\left(\left(\frac{-i}{2}\right)\int J(x) D(x-y) J(y) d^4x\, d^4y\right)\right)\Bigg|_{J=0}$$

with $\tilde{C} = Z(J=0)$ constant . (9')

For a Lagrangian density of fermion fields (quarks and leptons) interacting with gauge boson fields we have :

$$\begin{aligned}&\tilde{\mathscr{L}}((\psi^\alpha,\partial\psi^\alpha)_\alpha,(A^a,\partial A^a)_a) = \bar{\psi}^\alpha(i\,\delta_{\alpha\beta}\gamma^\mu\partial_\mu - m_\alpha\delta_{\alpha\beta})\psi^\beta + \\ &+\sum_g \Big(g\,\bar{\psi}^\alpha\gamma^\mu A_\mu^a T_{\alpha\beta}^a\psi^\beta - (1/4)(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - \\ &\quad -(1/2)g(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) f^{abc} A^{b\mu} A^{c\nu} - \\ &\quad -(1/4)g^2 f^{abc} f^{ade} A_\mu^b A_\nu^c A^{d\mu} A^{e\nu} + (1/2) M_a^2 A_\mu^a A^{a\mu}\Big)\end{aligned} \qquad (10)$$

In the electroweak SU(2)xU(1) or in the unified electroweak+chromodynamics SU(3)xSU(2)xU(1) theory for any g coupling we have a corresponding set of gauge bosons and respective gauge group generators defined coefficients :

$((A_\mu^a)_\mu,(T_{\alpha\beta}^a)_{\alpha\beta})_a$ with μ - Lorentz index

α,β - colour, flavour , lepton sort index

In quantum chromodynamics SU(3) or in the grand unified SU(5) theory we have an unique coupling constant g with the set of gauge bosons and gauge group generators.

We have following Feynman rules to compute Feynman amplitudes of fermion and gauge boson (gluon) particle transition processes for a perturbation theory approach (which is relevant in the case of a weak couplings like in electroweak theory or asymptotic freedom of quantum chromodynamics):
(The considered process has $q_1,\ldots,q_n$ incoming fermions momenta, $p_1,\ldots,p_s$ outgoing fermions momenta and $k_1,\ldots,k_h$ outgoing bosons momenta and the Feynman diagram is with

$x_1^a,\ldots,x_s^a$ outgoing fermions legs end vertices,
$x_1^b,\ldots,x_n^b$ incoming fermions legs end vertices,
$x_1,\ldots,x_r$ fermion interaction vertices,
$y_1,\ldots,y_k$ cubic gluon interaction vertices,
$z_1,\ldots,z_q$ quartic gluon interaction vertices,
$y_1^a,\ldots,y_h^a$ outgoing boson legs end vertices,
$(x_{1l},x_{2l})_{l=\overline{1,m}}$ internal lines).

1. For each interaction vertex write $(2\pi)^4\delta^4(\sum_{k\in A}k-\sum_{k\in B}k)$

(where A is the set of incoming to the vertex momenta and B is the set of outgoing from the vertex momenta)
and write the coupling: a) $ig\gamma^{\mu}$ for x_l vertices;
b) $gf^{abc}(\eta^{\mu\nu}(r_1-r_2)^{\lambda}+\eta^{\nu\lambda}(r_2-r_3)^{\mu}+\eta^{\lambda\mu}(r_3-r_1)^{\nu})$ where r_1,r_2,r_3 label the incoming to the cubic gluon interaction vertex respective a,b,c gluon momenta
(do not confuse the gluon indices a , b, c with the notations with a , b upper index for outgoing respective incoming legs) for y_l vertices.
c) $-ig^2\left(f^{abc}f^{ade}(\eta^{\mu\lambda}\eta^{\nu\epsilon}-\eta^{\mu\epsilon}\eta^{\nu\lambda})+f^{adc}f^{abe}(\eta^{\mu\lambda}\eta^{\nu\epsilon}-\eta^{\lambda\epsilon}\eta^{\mu\nu})+\right.$
$\left.+f^{abd}f^{ace}(\eta^{\nu\mu}\eta^{\lambda\epsilon}-\eta^{\nu\epsilon}\eta^{\mu\lambda})\right)$ for z_l vertices.

2. For each internal line write the propagator :

a) $\dfrac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$ for a mass m fermion line labeled with p momentum
b)
$i\left(-\eta_{\mu\nu}+\dfrac{k_\mu k_\nu}{M^2}\right)\dfrac{1}{k^2-M^2+i\varepsilon}$ for a mass M boson line labeled with k momentum.
c)For massless bosons we will consider a hypothetical tending to 0 non-vanishing mass in computations that are confronted with lattice method computations which will be further presented. In that case, the $k_\mu k_\nu/M^2$ term in the propagator disappears in computations because the masses of the two fermions linked in the amplitude expression to the propagator of a massless boson as the photon or the SU(3) bosons in SU(3)xSU(2)xU(1) theory or in quantum chromodynamics are

equal (flavour changing occurs only through the W bosons which are massive) and we can use (4') , (8') , (4'''). Otherwise we can have an additional ghost action by Fadeev-Popov method with gauge parameter leading to a propagator $i(-\eta_{\mu\nu}+(1-\xi)(k_\mu k_\nu/k^2))/k^2$ where we can take the gauge parameter $\xi=1$.

For example we have in an amplitude expression the factor

$$iD^{fer}_{\alpha\beta}(p)\gamma^\mu_{\beta\epsilon}iD^{bos}_{\mu\nu}(r)iD^{fer}_{\epsilon\delta}(q)\delta^4(p-q-r)$$ which contains

$$\bar{u}(p)\gamma^\mu u(q)(-\eta_{\mu\nu}+(r_\mu r_\nu)/M^2)\delta^4(p-q-r)=$$

$$=(-\bar{u}(p)\gamma_\nu u(q)+((p-q)_\nu(m-m)\bar{u}(p)u(q)/M^2))\delta^4(p-q-r)$$

and as we can see the term containing the hypothetical mass disappears since the involved fermions masses are equal.

3. Write $\bar{u}(p_j,s)$ for outgoing fermions, $u(q_j,s)$ for incoming fermions, write $\bar{v}(q_j,s)$ for incoming antifermions, $v(p_j,s)$ for outgoing antifermions, write $\varepsilon_\mu(k_j,s)$ for outgoing or incoming bosons.

4. Multiply the written factors and multiply the result with a (-1) factor for each closed fermion cycle.

5. Momenta k associated with internal lines are to be integrated over with $\frac{d^4k}{(2\pi)^4}$ measure.

6. The external legs are "amputated" since according to rule 2. we write the propagators only for internal lines. The particles are on mass shell (i.e. we have $p_j^2-m^2=0$, $q_j^2-m^2=0$, $k_j^2-M^2=0$ where m and M take the respective values of the corresponding particles).

The amplitude has the form $(2\pi)^4\boldsymbol{M}\,\delta^4(\sum_{j=1}^{s}p_j+\sum_{j=1}^{h}k_j-\sum_{j=1}^{n}q_j)$ with $\boldsymbol{M}$ an invariant Feynman amplitude.

Since the fermion field operators anti-commute, for a set $(\hat{\varphi}_i)_{i=\overline{1,m}}$ of operators in which $\hat{\varphi}_{i1},\ldots,\hat{\varphi}_{ir}$ with $i_1<i_2<\ldots<i_r$ anti-commute each with other and the remaining $\{1,\ldots,m\}\setminus\{i_1,\ldots,i_r\}\ni i,\hat{\varphi}_i$ operators commute with any of the operators in the set we define :

$$T(\prod_{i=1}^{m}\hat{\varphi}_i(x_i))=\sum_{\sigma\in Sm}\varepsilon(\tilde{\sigma})(\prod_{i=1}^{m-1}\theta(x^0_{\sigma(i)}-x^0_{\sigma(i+1)}))(\prod_{i=1}^{m}\hat{\varphi}_{\sigma(i)}(x_{\sigma(i)}))$$ where

$$\tilde{\sigma}=\sigma|_{\{i1,\ldots,ir\}}$$ and $$\varepsilon(\tilde{\sigma})=\operatorname{sgn}\prod_{1\le k<l\le r}(\sigma(i_l)-\sigma(i_k))$$

In the above definition for *T* we take $\theta(0)=1$ and divide rhe right member by $k!$ for every k times occurence of the same value of x_i^0.

Also, considering that for a fermion lines closed cycle with ($x_1,\ldots,x_{r+1}$) , $x_1=x_{r+1}$ interaction vertices , from

$\overline{\psi}(x_1)\,\gamma^{\mu} A_{\mu}(x_1)\,\psi(x_1)\,\overline{\psi}(x_2)\,\gamma^{\nu} A_{\nu}(x_2)\,\psi(x_2)\ldots\gamma^{\lambda} A_{\lambda}(x_r)\,\psi(x_r)$, in order to obtain the proper r vertices fermion cycle factor of the amplitude, $\mathrm{tr}\,(S(x_r-x_1)\,\gamma^{\mu} S(x_1-x_2)\,\gamma^{\nu}\ldots S(x_{r-1}-x_r)\,\gamma^{\lambda})$ we must anti-commute to $\psi_l(x_r)\,\overline{\psi}(x_1)\,\gamma^{\mu}\ldots\gamma^{\lambda} A_{\lambda}(x_r)$ (for the d^{+} of $\hat{\psi}(x_r)$ meet the d of $\hat{\overline{\psi}}(x_1)$)
(The x_i variables are obviously to be integrated over in the final amplitude expression). Thus we will have the extra (-) sign for each closed fermion cycle of the Feynman diagram and on the cycle we must have an anti-fermion propagating backwards in time.
To compute the total amplitude for the Feynman diagrams with the given outgoing and incoming momenta of fermions / anti-fermions and bosons and given numbers of fermion interaction vertices, cubic and quartic gluon interaction vertices we have to deal with the expression of amplitude *A* as follows :

$$A=cf\int\frac{1}{s!h!n!}\left\langle 0\left|\left(\sum_{\sigma\in Ss}\varepsilon(\sigma)\prod_{i=1}^{s}\tilde{b}(p_{\sigma(i)})\right)\sum_{\sigma\in Sh}\prod_{j=1}^{h}\tilde{a}(k_{\sigma(j)})T\,(R((x),(y),(z)))\right.\right.$$
$$\left.\left.\left(\sum_{\sigma\in Sn}\varepsilon(\sigma)\prod_{l=1}^{n}\tilde{b}^{+}(q_{\sigma(l)})\right)\right|0\right\rangle\left(\prod_{i=1}^{r}d^4x_i\right)\left(\prod_{i=1}^{k}d^4y_i\right)\left(\prod_{i=1}^{q}d^4z_i\right)$$

where *cf* is a coefficient from the exponential expansion carried by the interaction terms product we consider in the Feynman diagram and $R((x),(y),(z))$ has the form :

$$\left(\prod_{l=1}^{r}\hat{\overline{\psi}}^{\alpha}(x_l)\,i g\,\gamma^{\mu}\hat{A}^{a}_{\mu}(x_l)\,T^{a}_{\alpha\beta}\hat{\psi}^{\beta}\right)\left(\prod_{j=1}^{k}i\hat{K}_c(y_j)\right)\left(\prod_{j=1}^{q}i\hat{K}_q(z_j)\right)$$, where

$K_c(y_j)=-(1/2)g f^{abd}(\partial_{\mu}A^{a}_{\nu}-\partial_{\nu}A^{a}_{\mu})A^{b\mu}A^{d\nu}(y_j)$ and
$K_q(z_j)=-(1/4)g^2 f^{abc}f^{ade}A^{b}_{\mu}A^{c}_{\nu}A^{d\mu}A^{e\nu}(z_j)$.

and for normalization we have taken :

$$(\tilde{a},\tilde{b},\tilde{d})=\left(\frac{(2\pi)^3}{V}\right)^{1/2}(a,b,d)$$

Suppressing the spin and polarization indices in $u(p,s),u(q,s)$ and $\varepsilon(k,s)$, considering summation over them, we have:

$$A=cf\int\left\langle 0\left|T\left(\left(\prod_{j=1}^{s}\exp(i p_j x^{a}_{j})\overline{u}(p_j)\,\hat{\psi}(x^{a}_{j})\right)\left(\prod_{j=1}^{h}\exp(i k_j y^{a}_{j})(-\varepsilon^{\lambda}(k_j))\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\hat{A}_{\lambda}(y^{a}_{j})\right)\right)R((x),(y),(z))\,T\left(\prod_{j=1}^{n}\exp(-i q_j x^{b}_{j})u(q_j)\hat{\overline{\psi}}(x^{b}_{j})\right)\right|0\right\rangle \tag{11}$$
$$V^{-(s+h+n)/2}\left(\prod_{j=1}^{s}(E_{pj}/m)^{1/2}\right)\left(\prod_{j=1}^{h}(2\,\omega_{kj})^{1/2}\right)\left(\prod_{j=1}^{n}(E_{qj}/m)^{1/2}\right)$$
$$\left(\prod_{j=1}^{s}d^3\vec{x}^{a}_{j}\right)\left(\prod_{j=1}^{h}d^3\vec{y}^{a}_{j}\right)\left(\prod_{j=1}^{n}d^3\vec{x}^{b}_{j}\right)\left(\prod_{j=1}^{r}d^4x_j\right)\left(\prod_{j=1}^{k}d^4y_j\right)\left(\prod_{j=1}^{q}d^4z_j\right)$$

In (11) we have taken $x_i^{a0}=y_j^{a0}=T\,;x_l^{b0}=0$ and any space or time integration is on $[0,T]$ time interval and a volume V spatial domain.

We understand also that the incoming and outgoing fermions or bosons can have different boson sort or flavour indices which we have suppressed in the above expression.

The $\hat{\psi}$ and $\hat{A}$ operator functions in (10) are the same as given in (4c) and (4b) because we consider in the perturbation theory approach of Feynman diagrams the relations of type (3) with the not gauged free theory Lagrangian density from (10) (having g = 0).

Considering type (3) and (9') relations we will have (12) :

$$A=\sum_{\text{diagrams}}\tilde{C}S_F V^{-(s+h+n)/2}\int((\prod_{j=1}^{s}(E_{pj}/m)^{1/2}\exp(i\,p_j x_j^a)\bar{u}(p_j)iD^{fer}(x_j^a-x_{1lj}))$$
$$(\prod_{j=1}^{h}(2\,\omega_{kj})^{1/2}\exp(i\,k_j y_j^a)(-\varepsilon(k_j))iD^{bos}(y_j^a-y_{lj}))$$
$$(\prod_{j=1}^{n}(E_{qj}/m)^{1/2}\exp(-i\,q_j x_j^b)iD^{fer}(x_{2lj}-x_j^b)u(q_j))$$
$$(\prod(\text{internal lines propagators and couplings })))$$
$$(\prod_{j=1}^{r}d^4x_j)(\prod_{j=1}^{k}d^4y_j)(\prod_{j=1}^{q}d^4z_j)(\prod_{j=1}^{s}d^3\vec{x}_j^a)(\prod_{j=1}^{h}d^3\vec{y}_j^a)(\prod_{j=1}^{n}d^3\vec{x}_j^b)$$

with $\tilde{C}=Z(\eta=0,g=0)$ and S_F a symmetry factor.

As established, we integrate over $0<x_{1lj}^0<T=x_j^{a0}$ in the diagrams of the (12) sum and considering (4''') and (8'') we have:

$$\int(E_{pj}/m)^{1/2}\exp(i\,p_j x_j^a)\bar{u}(p_j)D^{fer}(x_j^a-x_{1lj})d^3\vec{x}_j^a= \qquad (13)$$
$$=(E_{pj}/m)^{-1/2}\bar{u}(p_j)\exp(i\,p_j x_{1lj})$$

and similar :

$$\int(E_{qj}/m)^{1/2}\exp(-i\,q_j x_j^b)iD^{fer}(x_{2lj}-x_j^b)u(q_j)d^3\vec{x}_j^b= \qquad (14)$$
$$=\exp(-i\,q_j x_{2lj})u(q_j)(E_{qj})^{-1/2}$$

and also we will have :

$$\int(2\,\omega_{kj})^{1/2}\exp(i\,k_j y_j^a)(-\varepsilon(k_j))iD^{bos}(y_j^a-y_{lj})d^3\vec{y}_j^a= \qquad (15)$$
$$=(2\,\omega_{kj})^{-1/2}\varepsilon(k_j)\exp(i\,k_j y_{lj})$$

To prove (15) we integrate over k^0 using the residues theorem in the boson propagator expression

$$\int\exp(-i\,k(y_j^a-y_{lj}))\frac{-\eta_{\mu\lambda}+(k_\mu k_\lambda/M^2)}{k^2-M^2+i\,\varepsilon}d^4k$$

and for that the integral over the semicircle $\{k^0=R\exp(i\,\theta)\}$, $\theta\in[-\pi,0]$ must be considered. The only case in which the integral not vanishes for $R\rightarrow\infty$ is when $\lambda=\mu=0$ and the remaining not vanishing term is $($ with $y=y_j^a-y_{lj})$:

$$\lim_{R\to\infty}\int_{-\pi}^{0}\frac{\exp(-iRe^{i\theta}y^0+i\vec{k}\vec{y})R^3\cos(3\theta)}{R^2e^{2i\theta}-\vec{k}^2-M^2+i\varepsilon}d\theta$$

Since $\cos(3\theta)=\cos(\theta)(1-2\sin^2(\theta))-2\sin^2(\theta)\cos(\theta)$ we will have the only not vanishing remaining term (after some calculus) :

$$B=\int_0^R\frac{\exp(-iR\sqrt{1-(x/R)^2}y^0+i\vec{k}\vec{y}-xy^0)}{R^2(1-2(x/R)^2)+2iRx\sqrt{1-(x/R)^2}-\vec{k}^2-M^2+i\varepsilon}R^2dx+$$
$$+\int_R^0\frac{\exp(iR\sqrt{1-(x/R)^2}y^0+\vec{k}\vec{y}-xy^0)}{R^2(1-2(x/R)^2)-2iRx\sqrt{1-(x/R)^2}-\vec{k}^2-M^2+i\varepsilon}R^2dx \qquad (16)$$

The integrands in (16) are dominated by the absolutely integrable function $\exp(-xy^0)$ for $x\in(0,\infty)$ and so taking a cut-off for integration over k with $|k^0|<R$ and large R we have $B=-2i\frac{\sin(Ry^0)}{y^0}\exp(i\vec{k}\vec{y})=P(y^0)\exp(i\vec{k}\vec{y})$

which integrated over $\vec{k}$ leads to $P(y^0)(2\pi)^3\delta^3(\vec{y}_j^a-\vec{y}_{lj})$.

In order to have external legs for the gauge bosons , we integrate the $\vec{y}_j^a$ and $\vec{y}_{lj}$ variables on a set $\|\vec{y}_j^a-\vec{y}_{lj}\|\geq\varepsilon$ and so,after $d^3\vec{y}_j^a d^4y_{lj}$ integration, the non-vanishing term left by applying the residues theorem on the boson propagator expression disappears and we can use (15) in the computation of the amplitude.

Note that for an outgoing anti-particle with momentum p_j we must take $\exp(ip_jx_j^a)\hat{\bar{\psi}}(x_j^a)v(p_j)$ instead of $\exp(ip_jx_j^a)\bar{u}(p_j)\hat{\psi}(x_j^a)$ and for an incoming antiparticle with momentum q_j we take $\exp(-iq_jx_j^b)\bar{v}(q_j)\hat{\psi}(x_j^b)$ instead of $\exp(-iq_jx_j^b)\hat{\bar{\psi}}(x_j^b)u(q_j)$ in the (11) expression.

The final and initial states of the considered process, which are $\langle 0|\tilde{b}(p_1)...\tilde{b}(p_s)\tilde{a}(k_1)...\tilde{a}(k_h)|=\langle 0|\psi_F$ and

$$\tilde{b}^+(q_1)...\tilde{b}^+(q_n)|0\rangle=|\psi_I\rangle$$

have to be normalized for computing the effective process amplitude *A* and the transition probability $|A|^2$ such that $\langle\psi_F|\psi_F\rangle=\langle\psi_I|\psi_I\rangle=1$.

We can prove that if $[R,\tilde{a}(q)]=0$ then $\langle 0|R\tilde{a}^l(q)\tilde{a}^{l+}(q)R^+|0\rangle=l!\langle 0|RR^+|0\rangle$.

Using this , and the fact that a state of many fermions of the same sort vanishes if there are two fermions with the same momentum (this is in fact the Pauli exclusion principle and follows from the anti-commutation relations) it follows that we must normalize with a factor of $1/\sqrt{l!}$ for each occurrence of l identical bosons with the same momentum and the corresponding transition probability will be adjusted by a statistical factor (eliminating double counting of events)

$S=\prod_i\frac{1}{l_i!}$ with l_i the number of occurrences of a boson with the same momentum.

Considering the way we calculate Feynman amplitudes (amputating external legs), the relations (12) , (13), (14), (15) and the symmetry factor that comes out to be the same from the many ways in which we can associate a Feynman diagram to an (11) expression we can conclude that the relation between the transition amplitude and the total amplitude for all Feynman diagrams of a given couplings order and given outgoing and incoming momenta of given respective fermions and respective bosons is :

$A=V^{-(s+h+n)/2}(\prod_{j=1}^{s}\left(\frac{E_{pj}}{m}\right)^{-1/2})(\prod_{j=1}^{n}\left(\frac{E_{qj}}{m}\right)^{-1/2})(\prod_{j=1}^{h}(2\omega_{kj})^{-1/2})A_F$ where

$A_F=(2\pi)^4\boldsymbol{M}\,\delta^4(\sum_1^s p_j+\sum_1^h k_i-\sum_1^n q_l)$ is the total Feynman amplitude.

The $u(p,s)$, $v(p,s)$, $\varepsilon(p,s)$ which are needed in the amplitude computation are determined by theirs normalization values in the rest frame.

For a decay process we have $n=1$ and since

$(\delta^4(q_1-\sum_1^h k_i-\sum_1^s p_j))^2=\frac{\delta^4(q_1-\sum_1^h k_i-\sum_1^s p_j)VT}{(2\pi)^4}$ and momentum space $d^3\vec{p}$

contains $\frac{V}{(2\pi)^3}d^3\vec{p}$ states, we can compute a differential decay rate $\frac{|A|^2}{T}$,

$$d\Gamma=\frac{(2\pi)^4 m}{E_{q1}}\left(\prod_{j=1}^{h}\frac{d^3\vec{k}_j}{(2\pi)^3 2\omega_{kj}}\right)\left(\prod_{j=1}^{s}\frac{m\,d^3\vec{p}_j}{(2\pi)^3 E_{pj}}\right)|\boldsymbol{M}|^2\delta^4(q_1-\sum_1^h k_i-\sum_1^s p_j)$$

For a two fermion scattering process we have $n=2$, $\vec{v}_1$, $\vec{v}_2$ velocities of the incoming particles, $\mathrm{n}=1/V$ concentration of a incoming particle, we compute a differential effective cross section $\frac{|A|^2}{T\,\mathrm{n}|\vec{v}_1-\vec{v}_2|}$, $d\sigma=$ (*)

$$\frac{(2\pi)^4 m_1 m_2}{|\vec{v}_1-\vec{v}_2|E_{q1}E_{q2}}\left(\prod_{j=1}^{h}\frac{d^3\vec{k}_j}{(2\pi)^3 2\omega_{kj}}\right)\left(\prod_{j=1}^{s}\frac{m\,d^3\vec{p}_j}{(2\pi)^3 E_{pj}}\right)|\boldsymbol{M}|^2\delta^4(\sum_1^2 q_l-\sum_1^h k_i-\sum_1^s p_j)$$

Obviously we have

$$((2\pi)^4\delta^4(p-p'))^2=(2\pi)^4\delta^4(p-p')\int\exp(-i(p-p')x)d^4x=$$
$$=(2\pi)^4\delta^4(p-p')VT$$ and so taking in discretization $\delta^4(p-p')=C\,\delta_{pp'}$

we obtain $\delta^4(p-p')=\frac{VT}{(2\pi)^4}\delta_{pp'}$

In the same way we have:

$\delta^3(\vec{p}-\vec{p}\,')=\frac{V}{(2\pi)^3}\delta_{\vec{p}\vec{p}'}$ and $((2\pi)^3\delta^3(\vec{p}-\vec{p}\,'))^2=(2\pi)^3\delta^3(\vec{p}-\vec{p}\,')V$

Also , to be able to count states, we enclose our system in a box, say a cube with length *L* on each side with *L* much larger than the characteristic size of our system, having $V = L^3$. With periodic boundary conditions , the allowed plane wave states $\exp(i\vec{p}\vec{x})$ carry momentum $\vec{p}=\frac{2\pi}{L}(n_1,n_2,n_3)$ where $n_i \in \mathbb{Z}$.

The allowed values of momentum form a lattice of points with spacing $2\pi/L$ between points. Experimentalists measure momentum with finite resolution, small but much larger than $2\pi/L$. Thus an infinitesimal volume $d^3\vec{p}$ in momentum space contains $d^3\vec{p}/(2\pi/L)^3 = V\,d^3\vec{p}/(2\pi)^3$ states.

In some cases we can split a process in small distance effects, as scattering $q_1,\dots,q_n$ incoming fermions momenta into $k_1,\dots,k_h$ outgoing bosons momenta and $p_1,\dots,p_s,q_{n+1},\dots,q_{n+r}$ outgoing fermions momenta, with $q_{n+1},\dots,q_{n+r}$ internal fermion lines momenta which in the large distance effects decay respective into external k_{h+i} boson momenta and p_{s+i} fermion momenta $(i=\overline{1,r})$

Since the $D^{fer}_{\alpha\beta}(q_{n+i})$ Fourier transform of the fermion propagator can be written as $\frac{2m\,u_\alpha(q_{n+i})\overline{u}_\beta(q_{n+i})}{q^2_{n+i}-m^2+i\varepsilon}$ and in the amplitude computation we must take $q_{n+i}=k_{h+i}+p_{s+i}$ we have the process amplitude factorization:

$$A_F(q,\widetilde{k},\widetilde{p})=A_F(q,k,\overline{p})\prod_{j=1}^{r}\frac{2mi}{(k_{h+j}+p_{s+j})^2-m^2+i\varepsilon}\boldsymbol{M}(k_{h+j}+p_{s+j},k_{h+j},p_{s+j})$$

where $q=(q_i)_{i=\overline{1,n}}$, $\widetilde{k}=(k_i)_{i=\overline{1,h+r}}$, $\widetilde{p}=(p_i)_{i=\overline{1,s+r}}$ and $\overline{p}=(p_1,\dots,p_s,k_{h+1}+p_{s+1},\dots,k_{h+r}+p_{s+r})$ and $A_F(a,b,c)$ is the Feynman amplitude for a incoming fermions, b outgoing bosons, c outgoing fermions momenta and $A_F=(2\pi)^4\boldsymbol{M}\,\delta^4((\sum a)-(\sum b)-(\sum c))$

We notice that the amplitude has a pike when the $k_{h+j}+p_{s+j}=q_{n+j}$ are on mass shell and so we can describe the transition probability of the process by the transition probability derived from the squared absolute value of the small distance effects amplitude which corresponds to an $(a,b,c)=(q,k,\overline{p})$ process.

In the case of quantum electrodynamics *U*(1) or electroweak *SU*(2)x*U*(1) theory, the renormalized couplings *g*, in a range of momentum are relatively small and so the higher order terms in *g* from the expansion of $\exp(i\int\mathscr{L}(\psi,\partial\psi,A,\partial A)d^4x)$ can be neglected, allowing a perturbation theory approach of the $q_1,\dots,q_n,k_1,\dots,k_h,p_1,\dots,p_s$ transition process, in which we take in consideration only the low order Feynman diagrams for the process.

In the case of quantum chromodynamics *SU*(3) , or unified *SU*(3)x*SU*(2)x*U*(1) or grand unified *SU*(5) theories the renormalized couplings go to zero when the momentum range goes to infinity and so we can have a perturbation theory approach only for a high momentum range (asymptotic freedom). For a lower momentum range we must take the amplitudes defined by following relation (17):

$$A=V^{-(s+h+n)/2}\int\left((\prod_{j=1}^{s}(E_{pj}/m)^{1/2}\exp(i\,p_j x_j^a))(\prod_{j=1}^{h}(2\,\omega_{kj})^{1/2}\exp(i\,k_j y_j^a))\right.$$
$$\left.(\prod_{j=1}^{n}(E_{qj}/m)^{1/2}\exp(-i\,q_j x_j^b))S((x^a),(y^a),(x^b))\right)(\prod_{j=1}^{s}d^3\vec{x}^a)(\prod_{j=1}^{h}d^3\vec{y}^a)(\prod_{j=1}^{n}d^3\vec{x}^b)$$

where

$$S((x^a),(y^a),(x^b))=\left\langle 0\left|T\left((\prod_{j=1}^{s}\overline{u}_\alpha(p_j)\hat{\psi}_\alpha(x_j^a))(\prod_{j=1}^{h}(-\varepsilon^\lambda(k_j))\hat{A}_\lambda(y_j^a))\right.\right.\right.$$
$$\left.\left.\left.(\prod_{j=1}^{n}\hat{\overline{\psi}}_\beta(x_j^b)u_\beta(q_j))\right)\right|0\right\rangle=(\prod_{j=1}^{s}\overline{u}_\alpha(p_j))(\prod_{j=1}^{h}(-\varepsilon^\lambda(k_j)))(\prod_{j=1}^{n}u_\beta(q_j))C\int D\,A\,D\,\psi D\,\overline{\psi}$$
$$\exp(i\int\mathscr{L}(\psi,\partial\psi,A,\partial A)d^4x)(\prod_{j=1}^{s}\psi_\alpha(x_j^a))(\prod_{j=1}^{h}A_\lambda(y_j^a))(\prod_{j=1}^{n}\overline{\psi}_\beta(x_j^b))$$

considering $\mathscr{L}(\psi,\partial\psi,A,\partial A)$ as in (10) with all interaction terms within and according to a type (3) relation C is a discretisation dependent constant.

Also we take $x_j^a=T\,,y_j^a=T\,,x_j^b=0$ with $[0,T]$ the interaction process time interval and V, the space volume for the fields interaction process.

Notice that the $\hat{\psi}$, $\hat{A}$ operators are no more defined by (4c) , (4b) relations, because we consider all Feynman diagrams associated with the process and take therefore all interaction terms products, which means that we consider the type (3) relation with the whole gauged Lagrangian density from (10).

Since the high order Feynman diagrams count (due to strong couplings), we expect that the quarks participate in interactions in groups (confinement) and so we have to consider that hadrons (groups of quarks and antiquarks of various colour indexes confined by gluon fields) will be forming.

The colour charge of a quark/antiquark defined by $(\psi_i)_{i=\overline{1,3}}$ with i colour index , ψ_i Dirac spinors,is defined by:

$\rho^a=\overline{\psi}_i\frac{1}{2}\lambda_{ij}^a\psi_j$ and there are 8 colour charges, one for each $(\lambda_{ij}^a)_{i,j}$ self-adjoint traceless 3x3 Gell-Mann matrix of the SU(3) colour gauge group generators:

$$\lambda^1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\ \lambda^2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\ \lambda^3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\ \lambda^4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},$$
$$\lambda^5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},\ \lambda^6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\ \lambda^7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\ \lambda^8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}.$$

The $\hat{\rho}^a$ observables with $\langle \varphi|\hat{\rho}^a|\psi\rangle=(1/2)\lambda^a_{ij}\langle\overline{\varphi}_i|\psi_j\rangle$ are hermitean but not commute, so we choose a maximal subset of commuting colour charge operators which is
$\{(1/2)\lambda_3,(1/2)\lambda_8\}$,to define a colour charge observable: $\hat{\rho}^3\vec{e}_3+\hat{\rho}^8\vec{e}_8$.

For $a=3,8$,if $\psi_c=\gamma^2\psi^*$ with ψ^* the complex conjugate of ψ , corresponds to the antiparticle to ψ we have: $\overline{\psi}_{ci}\lambda^a_{ij}\psi_{cj}=\psi_i^{\mathrm{T}}\gamma^{2+}\gamma^0\lambda^a_{ij}\gamma^2\psi^*_j=-\overline{\psi}_i\lambda^a_{ij}\psi_j$, because
$\lambda^{aT}=\lambda^a$ for $a=3,8$.
Therefore the antiquarks carry the opposite colour charges to the quarks colour charges.
The forming hadrons must be colour charge singlets and so they can be the tensorial products of wave functions as mesons (quark- antiquark pairs) :
$\psi_M(t,x_1,x_2)=\sum_{i=1}^{3}\frac{1}{\sqrt{3}}\psi_{ci}(t,x_1)\psi_i(t,x_2)$ or as three quark/antiquark baryons:

$$\psi_B(t,x_1,x_2,x_3)=\sum\frac{1}{\sqrt{6}}\epsilon_{ijk}\psi_i(t,x_1)\psi_j(t,x_2)\psi_k(t,x_3)$$

$$\psi_{cB}(t,x_1,x_2,x_3)=\sum\frac{1}{\sqrt{6}}\epsilon_{ijk}\psi_{ci}(t,x_1)\psi_{cj}(t,x_2)\psi_{ck}(t,x_3)$$

In the amplitude expressions they appear as
$\sum\overline{\psi}_i\gamma^\mu\psi_i=\psi^\mu_M$ vector meson,
$\sum\overline{\psi}_i\psi_i=\psi_M$ scalar meson,
$(\sum\epsilon_{ijk}\psi_{i\alpha}\psi_{j\beta}\psi_{k\gamma})=(\psi_B^{\alpha\beta\gamma})$ baryon,
$(\sum\epsilon_{ijk}\overline{\psi}_{i\alpha}\overline{\psi}_{j\beta}\overline{\psi}_{k\gamma})=(\psi_{\overline{B}}^{\alpha\beta\gamma})$ antibaryon

The common eigenvectors (colour eigenstates) of $\hat{\rho}^3$, $\hat{\rho}^8$ are
$\psi_r=\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes\psi=\vec{r}\otimes\psi$; $\psi_g=\begin{pmatrix}0\\1\\0\end{pmatrix}\otimes\psi=\vec{g}\otimes\psi$; $\psi_b=\begin{pmatrix}0\\0\\1\end{pmatrix}\otimes\psi=\vec{b}\otimes\psi$ with ψ a Dirac spinor function ,having colour charges respectively
$\vec{q}_r=\frac{1}{2}\vec{e}_3+\frac{1}{2\sqrt{3}}\vec{e}_8$; $\vec{q}_g=-\frac{1}{2}\vec{e}_3+\frac{1}{2\sqrt{3}}\vec{e}_8$; $\vec{q}_b=-\frac{1}{\sqrt{3}}\vec{e}_8$.

We have $\vec{q}_r+\vec{q}_g+\vec{q}_b=0.$ The mesons and baryons have neutral colour charge.
The mesons are integer spin particles (0 – scalar , 1 – vector) and the baryons are half integer spin particles.

For example the proton is $\sum\frac{1}{\sqrt{6}}\epsilon_{abc}u^a u^b d^c$ with a,b,c colour indices
u up-quark, d down-quark, where two of the quarks u^a,u^b,d^c carry opposite secondary spin quantum numbers (if the u^a , u^b , d^c are spin eigenvectors). The proton is a spin ½ particle. The same way spin ½ combination *udd* gives the other nucleon, known as the neutron.

For a three quark baryon for example, with $\widehat{\psi}^1, \widehat{\psi}^2, \widehat{\psi}^3$ respective the three quark operator functions, the process with p_1, p_2, p_3 outgoing four-momenta and q_1, q_2, q_3 incoming four-momenta, having $p_i = q_i, i = \overline{1,3}$ on mass shell will have an amplitude defined by the following relation (18):

$$A(\vec{p}_1, \vec{p}_2, \vec{p}_3) = \int \frac{1}{6} V^{-3} \left(\prod_{l=1}^{3} \exp(i p_l (x_l^a - x_l^b)) \overline{u}_{il\alpha l}^{l}(p_l) u_{jl\beta l}^{l}(p_l)(E_{pl}/m_l)\right)$$
$$\epsilon_{i1i2i3} \epsilon_{j1j2j3} \langle 0 | \widehat{\psi}_{i1\alpha 1}^1(x_1^a) \widehat{\psi}_{i2\alpha 2}^2(x_2^a) \widehat{\psi}_{i3\alpha 3}^3(x_3^a) \widehat{\overline{\psi}}_{j1\beta 1}^1(x_1^b) \widehat{\overline{\psi}}_{j2\beta 2}^2(x_2^b) \widehat{\overline{\psi}}_{j3\beta 3}^3(x_3^b) | 0 \rangle$$
$$\prod_{k=1}^{3} d^3 \vec{x}_k^a d^3 \vec{x}_k^b$$

where we use Einstein summation convention for $il, jl, \alpha l, \beta l$ indices and take $x_l^{a0} = T, x_l^{b0} = 0$, $p_l^0 = E_{pl} = \sqrt{\vec{p}_l^2 + m_l^2}$

il, jl are colour indices from 1 to 3 and $\alpha l, \beta l$ are Dirac indices from 0 to 3 .

Therefore, a energy eigenstate wave function $\psi_{0B} = \psi_{0B}(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ for the baryon can be derived, taking

$\psi_{0B} = \int (1/(2\pi)^9) \exp(i \vec{p}_l \vec{x}_l) A(\vec{p}_1, \vec{p}_2, \vec{p}_3) d^3 \vec{p}_1 d^3 \vec{p}_2 d^3 \vec{p}_3$ (19).

As we mentioned , for an antiparticle occurring instead of a particle in the composition of the baryon in the $A(p_1, p_2, p_3)$ expression we will take

$v_\alpha(p) \widehat{\overline{\psi}}_\alpha(x^a)$ instead of $\overline{u}_\alpha(p) \widehat{\psi}_\alpha(x^a)$ and

$\overline{v}_\beta(p) \widehat{\psi}_\beta(x^b)$ instead of $u_\beta(p) \widehat{\overline{\psi}}_\beta(x^b)$

Also we have:

$$\langle 0 | \widehat{\psi}_\alpha^1(x_1^a) \widehat{\psi}_\beta^2(x_2^a) \widehat{\psi}_\gamma^3(x_3^a) \widehat{\overline{\psi}}_\delta^1(x_1^b) \widehat{\overline{\psi}}_\varepsilon^2(x_2^b) \widehat{\overline{\psi}}_\varphi^3(x_3^b) | 0 \rangle =$$
$$= C \int DA D\psi D\overline{\psi} \Big(\exp(i \int \mathscr{L}(\psi, \partial\psi, A, \partial A) d^4 x) \quad (18')$$
$$\psi_\alpha^1(x_1^a) \psi_\beta^2(x_2^a) \psi_\gamma^3(x_3^a) \overline{\psi}_\delta^1(x_1^b) \overline{\psi}_\varepsilon^2(x_2^b) \overline{\psi}_\varphi^3(x_3^b) \Big)$$

It follows that for making computed theoretical predictions and comparisons of different processes , we must be able to compute (by making a suitable discretization) path integrals of the form

$\int DA D\psi D\overline{\psi} \exp(\mathscr{L}(\psi, \partial\psi, A, \partial A) d^4 x) O(A, \psi, \overline{\psi})$

where O is a function operator depending on the fields $A, \psi, \overline{\psi}$ and can be for example :

$$O(A, \psi, \overline{\psi}) = \int \Big(\left(\prod_{j=1}^{s} \exp(i p_j x_j^a)\right) \left(\prod_{j=1}^{h} \exp(i k_j y_j^a)\right) \left(\prod_{j=1}^{n} \exp(-i q_j x_j^b)\right) \quad (18'')$$
$$\left(\prod_{j=1}^{s} \psi_{\mu j}^{\alpha j}(x_j^a)\right) \left(\prod_{j=1}^{h} A_{\lambda j}^{aj}(y_j^a)\right) \left(\prod_{j=1}^{n} \overline{\psi}_{\nu j}^{\beta j}(x_j^b)\right) \Big) \left(\prod_{j=1}^{s} d^3 \vec{x}_j^a\right) \left(\prod_{j=1}^{h} d^3 \vec{y}_j^a\right) \left(\prod_{j=1}^{n} d^3 \vec{x}_j^b\right)$$

where $\alpha j, \beta j, a j$ are quark/lepton/gluon sort and colour indices and $\mu j, \nu j$ respective λj are Dirac and Lorentz indices.

Consider now a hadron with momentum $k=(k^0,\vec{k})$ and the light-cone coordinates $x^+=(x^0+\vec{x}\cdot\mathrm{vers}\,\vec{k})/\sqrt{2}$, $x^-=(x^0-\vec{x}\cdot\mathrm{vers}\,\vec{k})/\sqrt{2}$

$$\vec{x}_\perp=\vec{x}-(\vec{x}\cdot\mathrm{vers}\,\vec{k})\,\mathrm{vers}\,\vec{k}\ ,\ \vec{x}_\parallel=(\vec{x}\cdot\mathrm{vers}\,\vec{k})\,\mathrm{vers}\,\vec{k}\ .$$

The quark constituents of the hadron have momenta k_j (j = 1 , 2 for mesons and j = 1 , 2 , 3 for baryons) with fractions x_j and relations

$$k_j^+=x_j k^+\ ,\ \sum x_j=1\ ,\ \sum \vec{k}_{j\perp}=0\ ,\ 2k_j^+k_j^- -\vec{k}_{j\perp}^2=m_j^2\ ,\ \vec{k}_{j\perp}\cdot\mathrm{vers}\,k=0$$

$$x_j\in[0,1]\ ,\ k_j^0=\frac{1}{\sqrt{2}}\left(x_j k^+ +\frac{\vec{k}_{j\perp}^2+m_j^2}{2x_j k^+}\right)$$

$k^+k^-=M^2$ where M is the effective mass of the hadron,

$$k^0=\frac{1}{\sqrt{2}}\left(k^+ +\frac{M^2}{2k^+}\right)\ ,\ \vec{k}_j=\vec{k}_{j\perp}+\frac{1}{\sqrt{2}}\left(x_j k^+ -\frac{\vec{k}_{j\perp}^2+m_j^2}{2x_j k^+}\right)\mathrm{vers}\,\vec{k}\ .$$

Thus we have a functional dependence $k_j=k_j(x_j,\vec{k}_{j\perp},\vec{k})$ (20).

With the constituents momenta $(k_j)_j$ in the place of $(p_j)_j$ momenta in $(18),(19)$ like relations, we can change the variables $(k_j)_j$ to variables

$$((x_j)_{j=\overline{1,m-1}},(k_{j\perp}^1,k_{j\perp}^2)_{j=\overline{1,m-1}},\vec{k})=(\tilde{x},\tilde{k}_\perp,\vec{k})$$

where $k_{j\perp}^l=\vec{k}_{j\perp}\cdot e_l$, $e_l\cdot e_i=\delta_{il}$, $e_i\cdot\vec{k}=0$; $i,l=1,2$

and m is the hadron's number of quark constituents and so we have the hadron momentum space wave function and the hadron energy eigenstate wave function computable according to (18) , (18') respective (19) like relations in the form

$$A((k_j)_{j=\overline{1,m}})=\widetilde{DA}(\tilde{x},\tilde{k}_\perp,\vec{k})$$

$$\psi_{0H}((\vec{x}_j)_{j=\overline{1,m}})=\int B(\tilde{x},\tilde{k}_\perp,\vec{k},(\vec{x}_j)_{j=\overline{1,m}})\,d^{m-1}\tilde{x}\,d^{2m-2}\tilde{k}_\perp\,d^3\vec{k}\quad (21).$$

The distribution amplitude us defined as :

$DA(\tilde{x},\vec{k})=\int\widetilde{DA}(\tilde{x},\tilde{k}_\perp,\vec{k})\,d^{2m-2}\tilde{k}_\perp$ and taking $W=\int\left|DA(\tilde{x},\vec{k})\right|^2 d^{m-1}\tilde{x}$

we have that $\frac{1}{W}\left|DA(\tilde{x},\vec{k})\right|^2 d^{m-1}\tilde{x}$ describes the probability of finding the constituents in state of $(x_j)_{j=\overline{1,m}}$ fraction values of k^+ at hadron momentum $\vec{k}$.

(The location variables in (21), $\vec{x}_j$ and the momentum fractions x_j should obviously not be mixed up!)

With the relations (20), an amplitude for a process of (p_1 , … , p_s) outgoing fermions momenta, which are grouping themselves as outgoing hadrons (k_1 , … , k_m) having fraction values for constituents respectively

$((x_{il})_{l=\overline{1,mi}})_{i=\overline{1,m}}$ taking $\tilde{x}_i=(x_{il})_{l=\overline{1,mi-1}}$, $\tilde{x}=(\tilde{x}_i)_{i=\overline{1,m}}$,

(q_1 , … , q_n) incoming fermions momenta, which are grouping themselves as incoming hadrons(k'_1 , … , $k'_{m'}$) having fraction values for constituents respectively

$((x'_{il})_{l=\overline{1,m'i}})$ taking $\tilde{x}'_i=(x'_{il})_{l=\overline{1,m'i-1}}$, $\tilde{x}'=(\tilde{x}'_i)_{i=\overline{1,m'}}$,

(r_1 , … , r_h) outgoing bosons momenta , can be described as a function of the constituents fractions and normal momentum components for the hadrons , and of the momenta of the hadrons:

$$A(q,r,p)=A_H(\widetilde{x},\widetilde{k}_\perp,\widetilde{x}',\widetilde{k}'_\perp,\vec{k},\vec{k}',r)\quad .$$

To compute decay rates or cross sections we need the transition probabilities $|A|^2$.

Integrating $\left|A_H\right|^2$ over the $\widetilde{x},\widetilde{k}_\perp,\widetilde{x}'\ ,\ \widetilde{k}'_\perp$ variables with the weight

$$\left(\prod_{i=1}^{m}\frac{\left|\widetilde{DA}_i(\widetilde{x}_i,\widetilde{k}_{i\perp},\vec{k}_i)\right|^2}{W_i}\right)\left(\prod_{i=1}^{m'}\frac{\left|\widetilde{DA}'_i(\widetilde{x}'_i,\widetilde{k}'_{i\perp},\vec{k}'_i)\right|^2}{W'_i}\right)$$, where

$$W_i=\int\left|\widetilde{DA}_i(\widetilde{x}_i,\widetilde{k}_{i\perp},\vec{k}_i)\right|^2 d^{mi-1}\widetilde{x}_i d^{2mi-2}\widetilde{k}_{i\perp}$$

$$W'_i=\int\left|\widetilde{DA}'_i(\widetilde{x}'_i,\widetilde{k}'_{i\perp},\vec{k}_i)\right|^2 d^{m'i-1}\widetilde{x}'_i d^{2m'i-2}\widetilde{k}'_{i\perp}\ ,$$

we obtain a transition probability in terms of the momenta of the hadrons:

$$\left|\widetilde{A}_H\right|^2=\left|\widetilde{A}_H\right|^2(\vec{k},\vec{k}',r)\ .$$

Let for $i=\overline{1,m}$, $q^i=(q^i_l)_{l=\overline{1,mi}}$ the momenta of the quarks/antiquarks which are constituents of the outgoing hadron with four-momentum k_i .

As we noticed we have the bijective correspondence $(q^i_l)_l=(q^i_l)_l(\widetilde{x}_i,\widetilde{k}_{i\perp},\vec{k}_i)$.

The number of q^i states (on mass shell) corresponding to a hyper-volume $d\,w=d^{mi-1}\widetilde{x}_i d^{2mi-2}\widetilde{k}_{i\perp}d^3\vec{k}_i$ located at $(\widetilde{x}_i,\widetilde{k}_{i\perp},\vec{k}_i)$ is

$$d\,w=\left(\frac{V}{(2\pi)^3}\right)^{mi}\left|\det\frac{D((q^i_l)_l)}{D(\widetilde{x}_i,\widetilde{k}_{i\perp},\vec{k}_i)}\right|d^{mi-1}\widetilde{x}_i d^{2mi-2}\widetilde{k}_{i\perp}d^3\vec{k}_i$$

and therefore the number of k_i states (on mass shell) corresponding to a volume $d^3\vec{k}_i$ located at $\vec{k}_i$ is $\left(\frac{V}{(2\pi)^3}\right)^{mi}\widetilde{W}_i(\vec{k}_i)d^3\vec{k}_i$ where

$$\widetilde{W}_i(\vec{k}_i)=\int\left|\det\frac{D((q^i_l)_l)}{D(\widetilde{x}_i,\widetilde{k}_{i\perp},\vec{k}_i)}\right|d^{mi-1}\widetilde{x}_i d^{2mi-2}\widetilde{k}_{i\perp}$$

with integration on [0,1] for the momentum fractions variables and a certain bounded range of momentum for the normal momenta variables.

Thus we obtain computable differential decay rates and differential cross sections for a hadron decay or a two hadrons scattering to a number of outgoing hadrons processes:

$$d\Gamma=\frac{\left|\widetilde{A}_H\right|^2}{T}\prod_{i=1}^{m}\left(\frac{V}{(2\pi)^3}\right)^{mi}\widetilde{W}_i(\vec{k}_i)d^3\vec{k}_i\qquad(22)$$

$$d\sigma=\frac{\left|\widetilde{A}_H\right|^2}{\left|\vec{v}_1-\vec{v}_2\right|}\frac{V}{T}\prod_{i=1}^{m}\left(\frac{V}{(2\pi)^3}\right)^{mi}\widetilde{W}_i(\vec{k}_i)d^3\vec{k}_i\qquad(23)$$

where $\left|A_H\right|^2=\left|A_H\right|^2((\vec{k}_i)_{i=\overline{1,m}}\ ,\vec{k}'_1)$ for a hadron decay and $\left|A_H\right|^2=\left|A_H\right|^2((\vec{k}_i)_{i=\overline{1,m}}\ ,\vec{k}'_1,\vec{k}'_2)$ for a two hadrons scattering .

Obviously , also leptons or bosons can appear as outgoing particles. We simply include their momenta in the outgoing momenta list and do the calculations as they have no constituents as well and so if such a particle is listed under index j and so the list of its constituents is void and $m_j=0$.

k_i are the outgoing momenta, k'_i are the incoming momenta and $\vec{v}_1, \vec{v}_2$ are the velocities of the scattering hadrons V, T are spatial volume and respective time interval for the process action.
V, T are constants part of the lattice simulation we consider and the discretization and fermion Grassmann variables normalization constants which appear as coefficients in a lattice simulation computation are to be setup by measurements performed in one of any known physical process from the computing of which we can extract the coefficient and it will have the same value for any other process we further consider for computation.

Consider now the scattering process corresponding to the Feynman diagram in fig.1

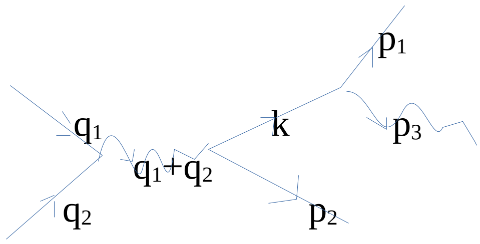


fig.1

q_1 , q_2 are the incoming fermions, p_1 , p_2 are outgoing fermions , p_3 is an outgoing boson four-momentum end legs lines labels and $q_1 + q_2$ labels as four-momentum an internal boson line, k labels as four-momentum an internal fermion line.
As we shown above we can factorize the fig.1 process through the decay of the k particle to p_1 and p_3 particles obtaining for the Feynman amplitudes the relation:

$$\widetilde{A}_F = A_F((q_1,q_2),(p_3),(p_1,p_2)) =$$
$$= A_F((q_1,q_2),\phi,(p_2,p_1+p_3)) \frac{2mi}{(p_1+p_3)^2 - m^2 + i\varepsilon} \boldsymbol{M}((p_1+p_3),(p_3),(p_1)) \quad (24)$$

where ϕ stands for an empty list of bosons four-momenta.
In the mass centre frame of the incoming particles (which are supposed to be on mass shell) we can consider
$\vec{q}_1 = (q, 0, 0)$, $\vec{q}_2 = (-q, 0, 0)$ and also $\vec{q}_i^2 = q_i^{02}$, $q_i^0 = q$ because we neglect the incoming fermions masses.
Momentum conservation leads to
$k = p_1 + p_3$, $\sum_i p_i^0 = 2q$, $\sum_i \vec{p}_i = \vec{q}_1 + \vec{q}_2 = 0$ and we take the fractions relations:
$p_i^0 = x_i q$, $\sum_i x_i = 2$ and neglecting fermion and boson masses we have also
$\|\vec{p}_i\| = x_i q$ since the particles are supposed to be on mass shell.
Let $\dfrac{\vec{p}_i \cdot \vec{p}_j}{\|\vec{p}_i\| \ \|\vec{p}_j\|} = \cos(\theta_{ij})$.

From the momentum conservation follows now

$$2(1-x_1)=x_2 x_3(1-\cos(\theta_{23}))$$
$$2(1-x_2)=x_1 x_3(1-\cos(\theta_{13}))$$
$$2(1-x_3)=x_1 x_2(1-\cos(\theta_{12}))$$

Using Feynman rules with the convention that Greek indices are Dirac spinor indices and Latin indices are fermion / boson designating indices , (24) becomes:

$$\tilde{A}_F=-(2\pi)^4 \frac{2m_c g^2}{4q^2(1-x_2)} \bar{u}^c_\alpha(p_1+p_3) T^a_{cd}\,\gamma^\mu_{\alpha\beta} v^d_\beta(p_2) \frac{1}{r^2}\left(\eta^{\mu\lambda}-\frac{r_\mu r_\lambda}{M_a^2}\right)\gamma^\lambda_{\rho\varepsilon}\bar{v}^{c'}_\rho(q_1) \quad (25)$$
$$u^{d'}_\varepsilon(q_2) T^a_{c'd'} u^c_\beta(p_1+p_3)\,\gamma^\nu_{\delta\beta}\bar{u}^e_\delta(p_1) T^{a'}_{ec}\varepsilon^{a'}_\nu(p_3)\,\delta^4(p_1+p_2+p_3-q_1-q_2)$$

where $r=q_1+q_2$.

Considering the (4') relations we have

$\bar{v}^{c'}_\rho(q_1)\,\gamma^\lambda_{\rho\varepsilon}(q_1+q_2)_\lambda u^{d'}_\varepsilon(q_2)=(-m_{c'}+m_{d'})\bar{v}^{c'}_\varepsilon(q_1)u^{d'}_\varepsilon$ and since we have taken $m_{c'}\approx m_{d'}\approx 0$ we can drop the $\frac{r_\mu r_\lambda}{M_a^2}$ term in (25) .

Taking $\tilde{A}_F=(2\pi)^4\tilde{\boldsymbol{M}}\,\delta^4(p_1+p_2+p_3-q_1-q_2)$ and considering (4) , (4''') relations, with summation over (averaged) spin polarizations, we will have

a $$|\tilde{\boldsymbol{M}}|^2=\frac{g^4}{16q^4(1-x_2)^2}\left|T^{a'}_{ec}T^a_{cd}T^a_{c'd'}\right|^2 4m_c^2\,\mathrm{tr}\left(\frac{\not p_1+\not p_3+m_c}{2m_c}\gamma^\mu\frac{\not p_2-m_d}{2m_d}\gamma^{\mu'}\right)$$
$$\mathrm{tr}\left(\frac{\not q_1-m_{c'}}{2m_{c'}}\gamma^\lambda\frac{\not q_2+m_{d'}}{2m_{d'}}\gamma^{\lambda'}\right)\frac{\eta^{\mu\lambda}\eta^{\mu'\lambda'}}{((q_1+q_2)^2)^2}\,\mathrm{tr}\left(\frac{\not p_1+m_e}{2m_e}\gamma^\nu\frac{\not p_1+\not p_3+m_c}{2m_c}\gamma^{\nu'}\right)$$
$$\left(-\eta^{\nu\nu'}+\frac{p_{3\nu}p_{3\nu'}}{M_{a'}^2}\right) \qquad (26)$$

We have

$$\mathrm{tr}\left(\frac{\not p_1+m_e}{2m_e}\gamma^\nu\frac{\not p_1+\not p_3+m_c}{2m_c}\gamma^{\nu'}\right)p_{3\nu}p_{3\nu'}=$$
$$=u^e_\alpha(p_1)\bar{u}^e_\beta(p_1)\not p_{3\beta\gamma}u^c_\gamma(p_1+p_3)\bar{u}^c_\delta(p_1+p_3)\not p_{3\delta\alpha} \text{ and}$$
$$\bar{u}^e(p_1)\not p_3 u^c(p_1+p_3)=(-m_e+m_c)\bar{u}^e(p_1)u^c(p_1+p_3)\approx 0$$

(since we take $m_e\approx m_c\approx 0$) .

Therefore we can drop the $\frac{p_{3\nu}p_{3\nu'}}{M_{a'}^2}$ term in the (26) expression for $|\tilde{\boldsymbol{M}}|^2$.

We can verify that

$\mathrm{tr}\,\gamma^\nu\gamma^\mu=4\eta^{\mu\nu}$, $\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma=4(\eta^{\mu\nu}\eta^{\lambda\sigma}-\eta^{\mu\lambda}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\lambda})$ and that the traces of a product of an odd number of gamma matrices vanish.

It follows :

$$\mathrm{tr}\left((\not{r}_1+m_1)\gamma^{\mu}(\not{r}_2+m_2)\gamma^{\nu}\right)=4\left(r_1^{\mu}r_2^{\nu}+r^{\nu}r_2^{\mu}-\eta^{\mu\nu}r_1\cdot r_2+\eta^{\mu\nu}m_1m_2\right)$$

and so for $B_{\lambda\lambda'}=\mathrm{tr}\left(\frac{\not{q}_1-m_{c'}}{2m_{c'}}\gamma^{\lambda}\frac{\not{q}_2+m_{d'}}{2m_{d'}}\gamma^{\lambda'}\right)$ we obtain

$$B_{\lambda\lambda'}=\begin{cases}\eta^{\lambda\lambda'}\left(1-\frac{2q^2}{m_{c'}m_{d'}}\right) & \text{for } \lambda,\lambda'\neq 0,1\\ 0 & \text{for } \{\lambda,\lambda'\}=\{0,1\}\\ \eta^{\lambda\lambda'} & \text{for } \lambda=\lambda'\in\{0,1\}\end{cases}\qquad(27)$$

and also, after some calculus:

$$\mathrm{tr}\left(\frac{\not{p}_1+m_e}{2m_e}\gamma^{\nu}\frac{\not{q}_1+\not{p}_3+m_c}{2m_c}\gamma^{\nu'}\right)\eta^{\nu\nu'}=4\left(1-\frac{q^2}{m_em_c}(1-x_2)\right)\qquad(28)$$

In the cross section expression we have , according to a (*) relation ,we will have a partial factor $\frac{m_{c'}m_{d'}}{q^2}$ and since we approximate $m_{c'}\approx m_{d'}\approx 0$, from the (27) factor, in the cross section expression we must keep only $-2\eta^{\lambda\lambda'}q^2$ with $\lambda,\lambda'=2,3$ having further:

$$\mathrm{tr}\left(\frac{\not{p}_1+\not{p}_3+m_c}{2m_c}\gamma^{\mu}\frac{\not{p}_2-m_d}{2m_d}\gamma^{\mu'}\right)(-2\eta^{\lambda\lambda'})\eta^{\mu\lambda}\eta^{\mu'\lambda'}=$$

$$=-2-\frac{1}{m_cm_d}\left(2p_{21}^2+2(2-x_2)x_2q^2\right)\qquad(29)$$

Since in the cross section expression we have also a partial factor $\frac{m_dm_e}{x_1x_2q^2}$ and we approximate $m_d\approx m_e\approx 0$ we must keep from the (28) factor only $-\frac{4}{m_c}(1-x_2)q^2$ and from the (29) factor, only $-2\left(p_{21}^2+(2-x_2)x_2q^2\right)$.

Thus the differential cross section is , after some calculus

$$d\sigma=\frac{f}{q^9}\frac{p_{21}^2+(2-x_2)x_2q^2}{(1-x_2)x_1x_2x_3}d^3\vec{p}_1d^3\vec{p}_2d^3\vec{p}_3\delta^4(q_1+q_2-p_1-p_2-p_3)=$$

$$=\frac{f}{q^7}\frac{x_2^2\cos^2(\theta_2)+(2-x_2)x_2}{(1-x_2)x_1x_2x_3}\sin(\theta_1)\sin(\theta_2)x_1^2x_2^2q^6dx_1dx_2d\theta_1d\theta_2d\varphi_1d\varphi_2$$

$$\delta(2q-q(x_1+x_2+x_3))d^3\vec{p}_3\delta^3(\vec{q}_1+\vec{q}_2-\vec{p}_1-\vec{p}_2-\vec{p}_3)$$

where $f=\left|T_{ec}^{a'}T_{cd}^{a}T_{c'd'}^{a}\right|^2\frac{1}{8(2\pi)^9}\frac{g^4}{|\vec{v}_1-\vec{v}_2|}$ (in the mass centre of the incoming particles frame).

Integrating over $(\theta_i,\varphi_i)\in(0,\pi)\times(0,2\pi)$, $i=1,2$ and $\vec{p}_3$ we obtain

$$d\sigma=\frac{32}{3}f\pi^2\frac{x_1x_2^2(3-x_2)}{q(1-x_2)(2-x-1-x_2)}\delta(2q-q(x_1+x_2+x_3))=$$
$$=F\frac{2x_2^2(3-x_2)}{x_3^2(1-\cos(\theta_{13}))}dx_1dx_2$$

where $F=\frac{16}{3}\frac{fT}{q}\pi$ and T is the process time interval .

(the Dirac distribution factor is over the 0 component of the four momentum, which is conjugated to time variable and we have therefore $\delta(q'-q'')=\frac{T}{2\pi}\delta_{q'q''}$)

The differential cross section has a pike at $x_3=0$ and at $\cos(\theta_{13})=1$.

In both cases it follows $x_2=1$ and $(p_1+p_3)^2=q^2((2-x_2)^2-x_2^2)=4q^2(1-x_2)=0.$
Therefore, since we neglected the fermion mass, the $k=p_1+p_3$ fermion can be considered on mass shell and $\vec{p}_1$, $\vec{p}_3$ are collinear.

Let us choose the x^3 axis close to the direction $\vec{k}=\vec{p}_1+\vec{p}_3$ and with orientation opposite to $\vec{p}_2$ orientation, so that the light-cone frame coordinates are

$$x^+=(x^3+x^0)/\sqrt{2}\ ,\ x^-=(x^0-x^3)/\sqrt{2}\ ,\ \vec{x}_\perp=(x_1,x_2,0)$$

Then $\vec{k}_\perp\approx\vec{0}$, $k^+=(p_1^0+p_3^0+\|\vec{p}_2\|)/\sqrt{2}=((2-x_2)q+x_2q)/\sqrt{2}=\sqrt{2}q$.
In the scattering experiments, 2 q is very large (it is the energy at which the particles collide in the mass centre frame).
Since the k particle is on mass shell when the cross section reaches the piked significant value, we have $k^-=(\vec{k}_\perp^2+m_c^2)/(2k^+)$.

Therefore, since $\vec{k}_{\perp,}\approx\vec{0}$, $m_c\approx 0$ and k^+ is very large, k^- must be very small and so $k^3=(k^+-k^-)/\sqrt{2}$ is also very large.
The k particle on mass shell propagates from the q_1+q_2 boson decay location $\vec{0}$ to the location $\vec{x}$ where decays into the p_1 fermion and the p_3 boson and because $\vec{k}_\perp\approx\vec{0}$ we can assume $\vec{x}_\perp\approx\vec{0}$

We have $\vec{k}=\frac{m_c\vec{v}}{\sqrt{1-\vec{v}^2}}$, the propagation time is $T=x^0$

and Lorentz invariance leads to $kx=m_cT\sqrt{1-\vec{v}^2}$. Also because $\vec{v}T=\vec{x}$

and $k^0=\frac{m_c}{\sqrt{1-\vec{v}^2}}$ we obtain $k^0x^3\approx k^3x^0$, $k^+x^-\approx k^-x^+$.

Hence $kx=k^+x^-+k^-x^+-\vec{k}_\perp\vec{x}_\perp\approx 2k^+x^-\approx 2k^-x^+$.
Because k^+ is very large and k^- is very small (obviously as an absolute value) it follows that $\vec{x}\approx(0,0,x^3)$ must be very large as an absolute value.
The scattering cross section goes to infinity when
$p_0^3\approx 0$ ($x_3=0$) and so we can call the fig.1 diagram not infrared safe.
Since x^3 is very large, the decay of the k emergent particle into a p_1 fermion and a p_3 boson occurs at a large distance from the q_1,q_2 fermions scattering point and

therefore we can reduce a scattering process through factorization, as exposed, to the small distance effects (in the fig.1 case the scattering to *k* and p_2 fermions on mass shell) which will be infrared safe.

Consider now a quark-antiquark meson. The constituents quark and antiquark constantly change colour due to strong interaction such that when a colour α quark is at location $\vec{r}_1$ an anticolour α antiquark is at location $\vec{r}_2$ with $\alpha \in \{\text{r,g,b}\}$. For the quark-gluon-antiquark interaction within the meson, the gluon fields change much faster than the quark and antiquark fields an so we can consider a potential energy of the quark-antiquark pair which is $V(\vec{r}) = E(r)$ where $\vec{r} = \vec{r}_1 - \vec{r}_2$, $r = \|\vec{r}\|$ and $E(r)$ is the energy of the gluons intermediating the quark-antiquark interaction.
During the gluon fields interaction time *T*, while the quark and antiquark are respectively at location $\vec{r}_1$ and $\vec{r}_2$ we have a quark colour charge current $J_1^{a\mu}(t,\vec{x}) = g\,\psi^{\alpha T} T^a_{\alpha\beta}\psi^\beta \overline{\psi}_1 \gamma^\mu \psi_1(t,\vec{x})$ and an antiquark colour charge current $J_2^{a\mu}(t,\vec{x}) = -g\,\psi^{\alpha T} T^a_{\alpha\beta}\psi^\beta \overline{\psi}_{2c} \gamma^\mu \psi_{2c}(t,\vec{x})$, where we have a minus sign since the quark and antiquark carry opposite colour charges and the notations correspond to : $T^a = \frac{1}{2}\lambda^a$, the ψ^α is one of the three colour charge eigenvectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $\psi_c = \gamma^2 \psi^*$ noticing that $\overline{\psi}_c \gamma^\mu \psi_c = \overline{\psi} \gamma^\mu \psi$ with ψ_i a Dirac spinor.
Considering the location of the quark and antiquark during the faster changing gluon fields intermediated interaction we can take $(\overline{\psi}_1 \gamma^\mu \psi_1)_\mu = (\delta^3(\vec{x} - \vec{r}_1), 0, 0, 0)$ and $(\overline{\psi}_2 \gamma^\mu \psi_2)_\mu = (\delta^3(\vec{x} - \vec{r}_2), 0, 0, 0)$.
Not considering the cubic and quartic gluon interactions the gluon fields Lagrangian density is :

$$\mathscr{L}((A^a, \partial A^a)_a) = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) + \frac{1}{2} M_a^2 A^a_\mu A^{a\mu} + (J_1^{a\mu} + J_2^{a\mu}) A^a_\mu$$

We have :

$$Z(J) = \exp(-iE(r)T) = Z(J=0)\exp\left(-(i/2)\int J^a(x) D^a(x-y) J^a(y)\, d^4x\, d^4y\right)$$

where $D^a_{\mu\nu}(x-y) = \int -\frac{1}{(2\pi)^4}\frac{\exp(-ik(x-y))}{k^2 - M_a^2 + i\varepsilon}\left(\eta^{\mu\nu} - \frac{k_\mu k_\nu}{M_a^2}\right) d^4k$ is the gluon propagator.
Excluding the vacuum energy (that is excluding *Z* (*J* = 0)) we can take

$$E(r)T = \int \left(\frac{1}{2}(J_1^a(x) D^a(x-y) J_1^a(y) + J_2^a(x) D^a(x-y) J_2^a(y)) + \right.$$

$$\left. + J_1^a D^a(x-y) J_2^a(y)\right) d^4x\, d^4y = \int d^4k \int dx^0\, dy^0 \exp(-ik^0(x^0 - y^0)) \qquad (30)$$

$$\frac{-1 + k^{02}/M_a^2}{k^2 - M_a^2 + i\varepsilon} S_{\alpha a} \frac{g^2}{(2\pi)^4}(1 - \exp(\vec{k}\vec{r})) = T\frac{g^2}{(2\pi)^3} S_{\alpha a} \int \left(\frac{1 - \exp(i\vec{k}\vec{r})}{\vec{k}^2 + M_a^2}\right) d^3\vec{k}$$

where $S_{\alpha a}=0$ for $a\notin\{3,8\}$, $S_{\alpha 3}=\frac{1}{4}$, $S_{\alpha 8}=\frac{1}{12}$ for $\alpha\in\{\mathrm{r,g}\}$, $S_{b3}=0$, $S_{b8}=\frac{1}{3}$ and in (30) we take the summation over *a* index.

Taking $M_3=M_8=0$ we have:

$$E(r)=E_0-\frac{g^2}{3(2\pi)^3}\int\frac{\exp(i\vec{k}\vec{r})}{\vec{k}^2}d^3\vec{k}$$

$$\int\frac{\exp(i\vec{k}\vec{r})}{\vec{k}^2}d^3\vec{k}=2\pi\int\int_0^{\pi}\exp(ikr\cos(\theta))\sin(\theta)d\theta dk=4\pi\int_0^{\infty}\frac{\sin(kr)}{kr}dk$$

We integrate over a range of momentum $k=\|\vec{k}\|$ for which $kr\ll 1$ so that we have quark confinement (the SU(3) chromodynamics coupling is strong at low energy).

Let $\|\vec{k}\|<a$. Hence with $ar\ll 1$ we will have:

$$E(r)=E_0-\frac{g^2}{3(2\pi)^3}4\pi\int_0^{ar}\frac{1}{r\tau}\sin(\tau)d\tau\approx$$

$$\approx E_0-\frac{g^2}{6\pi^2 r}\int_0^{ar}\left(1-\frac{1}{6}\tau^2\right)d\tau=E_0-g^2\frac{a}{6\pi^2}+Br^2$$

where $B=\frac{g^2a^3}{108\pi^2}$ and we take $V(\vec{r})=Br^2$ the potential energy of the quark-antiquark system.

The wave function of the meson, $\psi(t,\vec{r}_1,\vec{r}_2)=\exp(-i\hat{H}t)\,\psi_M(\vec{r}_1,\vec{r}_2)$ taking m as the effective mass of the meson, satisfies the time-independent Schroedinger equation :

$$E\,\psi_M(\vec{r}_1,\vec{r}_2)=-\frac{1}{2m}\nabla^2_{\vec{r}_1,\vec{r}_2}\psi_M(\vec{r}_1,\vec{r}_2)+V(\vec{r})\,\psi_M(\vec{r}_1,\vec{r}_2)\qquad(31)$$

where E is the energy level of the meson.

Searching for $\psi_M(\vec{r}_1,\vec{r}_2)=\frac{1}{m}x^{-3/4}F(x)$, $G(x)=F(bx)$, $x=\|\vec{r}_1-\vec{r}_2\|^2$

te equation (31) becomes $\frac{d^2G}{dx^2}(x)+\left(-\frac{mBb^2}{4}+\frac{mbE}{4x}+\frac{3}{16x^2}\right)G(x)=0$ (32)

We choose b such that $mBb^2=1$ and take $\kappa=\frac{mbE}{4}$, $\mu=\frac{1}{4}$ and so the (32) equation for G is the Whittaker function equation

$$\frac{d^2G}{dx^2}+\left(-\frac{1}{4}+\frac{\kappa}{x}+\frac{1/4-\mu^2}{x^2}\right)G=0\qquad(33).$$

The equation (33) , with parameters κ , μ has a fundamental system of solutions $M_{\kappa,\mu}$, $W_{\kappa,\mu}$

$$M_{\kappa,\mu}(z)=z^{\frac{1}{2}+\mu}\exp(-\frac{1}{2}z)\left(1+\sum_{p=1}^{\infty}\frac{\left(\frac{1}{2}+\mu-\kappa\right)\dots\left(\frac{1}{2}+\mu-\kappa+p-1\right)}{p!(2\mu+1)\dots(2\mu+p)}z^p\right)$$

$$W_{\kappa,\mu}(z)=\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\kappa)}M_{\kappa,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\kappa)}M_{\kappa,-\mu}(z)$$

For $\kappa=\mu-\frac{1}{2}+n$, $n\in\mathbb{N}^*$ we have that $M_{\kappa,\mu}(z)=z^{\frac{1}{2}}\exp(-\frac{1}{2}z)P(z)$ where P is a polynomial of degree $n-1$.

Therefore, for energy levels E_n , $E_n=(4n-1)\sqrt{\frac{B}{m}}$, $n\in\mathbb{N}^*$ the energy eigenstates are polynomial defined by the relations :

$$\psi_{Mn}(\vec{r}_1,\vec{r}_2)=\frac{1}{m}x^{-3/4}M_{\kappa n,1/4}(x/b)=$$

$$=\frac{b^{-3/4}}{m}\exp(-\frac{x}{2b})\left(1+\sum_{p=1}^{n-1}(-1)^p\frac{(n-1)\dots(n-p)}{p!1\cdot 3\dots(2p+1)}2^p\left(\frac{x}{b}\right)^p\right) \qquad (34)$$

$$\kappa_n=n-\frac{1}{4}\ ,\ x=\|\vec{r}_1-\vec{r}_2\|\ ,\ b=(mB)^{-1/2}$$

Since a wave function $\psi_{0M}(\vec{r}_1,\vec{r}_2)$ is computable for the meson in a lattice simulation (as in (19) for the baryon example) equating this function with the (34) relation function we should be able to determine the constants B , b , g^2a^3in the range of momentum given by a.

Consider now a three quark baryon consisting of three quarks with masses m_1, m_2 , m_3 and having different colours at a time.
As above , in this case we will have three colour charge currents

$$J_1^{a\mu}(t,\vec{x})=g\,\delta_{r\alpha}\frac{1}{2}\lambda_{\alpha\beta}^a\delta_{r\beta}\delta^3(\vec{x}-\vec{r}_1)\delta_{\mu 0}$$

$$J_2^{a\mu}(t,\vec{x})=g\,\delta_{g\alpha}\frac{1}{2}\lambda_{\alpha\beta}^a\delta_{g\beta}\delta^3(\vec{x}-\vec{r}_2)\delta_{\mu 0}$$

$$J_3^{a\mu}(t,\vec{x})=g\,\delta_{b\alpha}\frac{1}{2}\lambda_{\alpha\beta}^a\delta_{b\beta}\delta^3(\vec{x}-\vec{r}_3)\delta_{\mu 0}$$

where $\vec{r}_1,\vec{r}_2,\vec{r}_3$ are the position vectors of the three quarks during the faster changing gluon fields intermediated interaction in which we must consider all possible permutation of colour index values over the 1 , 2 , 3 positions in the interaction time interval of length T.

For $d_1=\|\vec{r}_2-\vec{r}_3\|$, $d_2=\|\vec{r}_3-\vec{r}_1\|$, $d_3=\|\vec{r}_1-\vec{r}_2\|$, the potential energy of the three quark system is $V(\vec{r}_1,\vec{r}_2,\vec{r}_3)=E(d_1,d_2,d_3)$ and satisfies:

$$E\,T = E_0{}'T + \sum_a \int \left(J_1^a(x) D^a(x-y) J_2^a(y) + J_2^a(x) D^a(x-y) J_3^a(y) + \right.$$
$$\left. + J_3^a(x) D^a(x-y) J_1^a(y) \right) d^4x\, d^4y$$

where in the sum over *a* we consider an average over all permutations of the colour index values over the 1 , 2, 3 positions.

Following steps as in the calculation for the meson case it follows that we can take

$V(\vec{r}_1, \vec{r}_2, \vec{r}_3) = B \sum_{i=1}^{3} d_i^2$ with $B = \frac{g^2 a^3}{216\pi^2}$ for $a\, d_i \ll 1$, a the range of momentum.

The energy eigenstates of the baryon system satisfy the time independent Schroedinger equation :

$$E\,\psi_B(\vec{r}_1, \vec{r}_2, \vec{r}_3) = V(\vec{r}_1, \vec{r}_2, \vec{r}_3)\,\psi_B(\vec{r}_1, \vec{r}_2, \vec{r}_3) + \sum_{i=1}^{3} -\frac{1}{2m_i} \nabla_{\vec{r}_i}^2 \psi_B(\vec{r}_1, \vec{r}_2, \vec{r}_3)$$

which for $x_i = d_i^2$, $i = \overline{1,3}$, $\psi_B(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \psi(x_1, x_2, x_3)$ becomes:

$$E\,\psi = (x_1 + x_2 + x_3) B\,\psi + \sum \frac{1}{m_1} \left(2 \frac{\partial^2 \psi}{\partial x_2^2} x_2 + 2 \frac{\partial^2 \psi}{\partial x_3^2} x_3 + 3 \frac{\partial \psi}{\partial x_2} + 3 \frac{\partial \psi}{\partial x_3} \right)$$

where the sum is taken over all circular permutations of (1 , 2 , 3).

We have solutions in the form $\psi(x_1, x_2, x_3) = \psi_1(x_1)\,\psi_2(x_2)\,\psi_3(x_3)$,

$$E = E_1 + E_2 + E_3 \text{ with } E_i\,\psi_i(x) = -\frac{1}{\overline{m}_i}(2\,\psi''(x)\,x + 3\,\psi_i(x)) + B\,\psi_i(x)\,x \quad (35)$$

where $\frac{1}{\overline{m}_1} = \frac{1}{m_2} + \frac{1}{m_3}$ with circular permutations over (1, 2, 3)

In the same way as for the meson wave function we obtain polynomial defined solutions

$$\psi_{in}(x) = \frac{b_i^{-3/4}}{\overline{m}_i} \exp\left(-\frac{x}{2b_i}\right) \left(1 + \sum_{p=1}^{n-1} (-1)^p \frac{(n-1)\ldots(n-p)}{p!\,1\cdot 3\ldots\cdot(2p+1)} 2^p \left(\frac{x}{b_i}\right)^p \right)$$

with $b_i = (2\overline{m}_i B)^{-1/2}$ for partial energy level $E_{in} = (4n-1)\sqrt{\frac{B}{2\overline{m}_i}}$.

The corresponding energy levels are $E_{n1n2n3} = E_{1n1} + E_{2n2} + E_{3n3}$ with eigenstates defined by $\psi_{n1n2n3}(x_1, x_2, x_3) = \psi_{n1}(x_1)\,\psi_{n2}(x_2)\,\psi_{n3}(x_3)$, $n_i \in \mathbb{N}^*$, $i = \overline{1,3}$.

So we have the (lowest level) eigenstate ψ_{0B} and with (19) we can recover the hadron momentum space wave function, needed in distribution amplitude calculations, by a Fourier transform.

As we mentioned we must be able to compute path integrals having the form

$$\int DA\,D\psi\,D\overline{\psi} \exp\left(i \int \mathscr{L}(\psi, \partial\psi, A, \partial A)\, d^4x\right) O(\psi, \overline{\psi}) \quad (36)$$

where $O(\psi, \overline{\psi})$ can have for example the expression:

$$O(\psi, \overline{\psi}) = \left(\prod_{i=1}^{s} \psi_{\beta i}^{\alpha i}(x_i)\right)\left(\prod_{j=1}^{n} \overline{\psi}_{\delta j}^{\gamma j}(x_j{}')\right) \text{ with}$$

$\alpha i, \gamma j$ colour and fermion sort indices and $\beta i, \delta j$ Dirac spinor indices.

Since the Lagrangian density $\mathscr{L}$ has a expression like in (10) , the path integral over $D\psi D\overline{\psi}$, where $\psi, \overline{\psi}$ can be considered independent sets of Grassmann variables, can be computed as a sum of Wick contraction terms, as shown for the (9') relation (with $\eta,\overline{\eta}$ variables conjugated to $\overline{\psi}$ respective ψ on the sides of the propagator, which is *D* (*x*-*y*) in the (9') relation) and so for the (36) integral not vanish ,we must have *s* = *n* .

To compute (36) we perform first a Wick rotation to imaginary time $t \rightarrow it = t_E$ and formulate the theory on a hyper-cubic lattice in 4-dimensional (Wick rotated Minkowski space-time $(t,\vec{x}) \rightarrow (t_E,\vec{x})$) Euclidean space-time, Λ= $\{(n_\mu a)_\mu\}_{n_\mu \in \mathbb{Z}\ ,\ \mu=\overline{0,3}}$, $\psi(t,\vec{x})=0$ if $|t|>T$ or exists $k \in \{1,2,3\}$ such that $|x^k|>L$.

As the lattice spacing *a* goes to 0, we expect to recover 4-dimensional rotational invariance and (by Wick rotation) Lorentz invariance.

The relativistic relation

$E^2/c^2-\vec{p}^2=m^2c^2$ with E energy, $\vec{p}$ momentum, m rest mass becomes by Wick rotation to imaginary time:

$$E'^2/c'^2-\vec{p}'^2=m^2c'^2 \text{ with } c'=-ic.$$

For $\hbar=1$, $c=1$ and $E'=i\frac{\partial}{\partial it}$, $p'_k=-i\frac{\partial}{\partial x^k}$ as translation generators,

we will have $-\partial^2/\partial t_E^2-\partial_k\partial_k=m^2.$

Therefore the corresponding Dirac equation for the Wick rotated space-time must be $\left(\gamma^0\frac{\partial}{\partial t_E}+i\gamma^k\partial_k-im\right)\psi=0$ and the Euclidean Lagrangian for a free fermion theory is

$\mathscr{L}_E(\psi,\partial\psi)=\overline{\psi}\left(i\gamma^0\frac{\partial}{\partial t_E}+\gamma_k\partial_k+m\right)\psi$ and so $\exp(i\int\mathscr{L}(\psi,\partial\psi)d^4x)$ which occurs in the theory path integral formalism, becomes in the Wick rotated space-time $\exp(i\int\mathscr{L}_E(\psi,\partial\psi)d\,it\,d^3\vec{x})=\exp(-S_E(\psi,\overline{\psi}))$ where

$S_E(\psi,\overline{\psi})=\int\overline{\psi}\left(i\gamma^0\frac{\partial}{\partial it}+\gamma_k\partial_k+m\right)\psi d\,t\,d^3\vec{x}=\int\overline{\psi}(\gamma_\mu\partial_\mu+m)\psi d\,t\,d^3\vec{x}$ is the euclidean action.

On each link, say the one going from $x\in\Lambda$ to one of its nearest neighbours $x+a\hat{\mu}\in\Lambda$ where $\hat{\mu}=(\delta_{\alpha\mu})_{\alpha=\overline{0,3}}$ we associate an *N* by *N* unitary simple matrix, parallel transporter $U_\mu(x)\in SU(N)$ with N the number of colour x flavour/ lepton sort indices:

$$U_\mu(x)=\exp(-i\int_x^{x+a\hat{\mu}}\sum_g gA_\mu^bT^b dx^\mu)$$

the T^b are $N\times N$ hermitian traceless matrices, $(A_\mu^b)_b$ are real gauge boson fields which we can normalize such that $\mathrm{tr}(T^cT^b)=\frac{1}{2}\delta_{cb}$, $\mathrm{tr}\,T^b=0$.

the $(T^b)_b$ are the generators of the gauge group representation. For each g coupling we have a set $(A^a,T^a)_a$ of gauge bosons and generators .

We have $U_\mu(x)=\mathbf{I}-i\sum_g agA_\mu^b(x)T^b+O(a^2)$

Obviously we used Einstein summation convention for the *b* index.
Considering the form of the euclidean free fermion theory in a fermions interacting gauged theory we will take a discretized euclidean fermion action

$$S_F(\psi,\overline{\psi})=a^4\sum_{x\in\Lambda}\overline{\psi}(x)(\not{D}+m)\psi(x)$$

where $\psi=(\psi^{\alpha})_{\alpha}$, ψ^{α} Dirac spinor, $m=\mathrm{diag}(m_{\alpha})_{\alpha}$, α colour x flavour/lepton sort index , m_{α} mass of the α fermion ,

$$\not{D}=\gamma_{\mu}(\partial_{\mu}-i\sum_{g}g\,A_{\mu}^{b}(x)T^{b})$$

(where obviously we used the discretization of the ∂_{μ} operator)

$$\partial_{\mu}f(x)=\frac{f(x+a\hat{\mu})-f(x)}{a} \ .$$

We have also the gluon fields discretized euclidean action $S_G(U)$ we must establish.
Consider the square $P(x)$, known as a plaquette, bounded b the corners x , $x+a\hat{\mu}$, $x+a\hat{\mu}+a\hat{\nu}$, $x+a\hat{\nu}$ with $x\in\Lambda$.
For each plaquette $P(x)$ we consider the expression:

$$P_{\mu\nu}(x)=U_{\mu}(x)U_{\nu}(x+a\hat{\mu})U_{\mu}^{+}(x+a\hat{\nu})U_{\nu}^{+}(x) \ .$$

Since $[T^b,T^c]=i f^{dbc}T^d$, $\mathrm{tr}(T^bT^c)=\frac{1}{2}\delta_{bc}$, $\mathrm{tr}\,T^b=0$ we have :

$$\mathrm{tr}\,P_{\mu\nu}=\mathrm{tr}\,\exp(-i a^2\sum_{g}F_{\mu\nu}^{b}T^{b}+O(a^3))$$

where $F_{\mu\nu}^{b}=g(\partial_{\mu}A_{\nu}^{b}-\partial_{\nu}A_{\mu}^{b})+g^2 f^{bcd}A_{\mu}^{c}A_{\nu}^{d}$
Under a gauge transformation $\psi(x)\rightarrow\Omega(x)\psi(x)$, $\Omega(x)\in SU(N)$,
the U_{μ} fields transform like $U_{\mu}(x)\rightarrow\Omega(x)U_{\mu}(x)\Omega^{+}(x+a\hat{\mu})$
We take the lattice plaquette gauge invariant euclidean action

$$S(P)=\sum_{g}\sum_{\mu\nu}(1/g^2)\Re\,\mathrm{tr}(\mathbf{I}-P_{\mu\nu})=(1/4)\sum_{g}(a^4/g^2)\Re\,\mathrm{tr}(F_{\mu\nu}^{b}F_{\mu\nu}^{b})+O(a^6)$$

The lattice euclidean gluon fields discretized action will be:

$$S_G[U]=\sum_{x\in\Lambda}(S(P)(x)+a^4\sum_{\mu,b}M_b^2A_{\mu}^{b2}(x))$$

, where M_b is the mass of the b boson and $A_{\mu}^{b}(x)=(2/(ga))\mathrm{tr}(i(U_{\mu}(x)-\mathbf{I})T^{b})$.

Further we will take $\nabla_{\mu}^{s}\psi(x)=\dfrac{U_{\mu}(x)\psi(x+a\hat{\mu})-U_{\mu}^{+}(x)\psi(x-a\hat{\mu})}{2a}$

and under a gauge transformation $\psi(x)\rightarrow\Omega(x)\psi(x)$ it will folow

$$\nabla_{\mu}^{s}\psi(x)\rightarrow\Omega(x)\nabla_{\mu}^{s}\psi(x)+O(a) \ .$$

We have also :

$$\nabla_{\mu}^{s}\psi(x)=\frac{U_{\mu}(x)+U_{\mu}^{+}}{2}\partial_{\mu}\psi(x)+\frac{U_{\mu}(x)-U_{\mu}^{+}(x)}{2a}\psi(x)=$$

$$=(\partial_{\mu}-i\sum_{g}g\,A_{\mu}^{b}(x)T^{b})\psi(x)+O(a)$$

Taking $S_F[U](\psi,\overline{\psi})=a^4\sum_{x\in\Lambda}\overline{\psi}(x)(\gamma_\mu\nabla_\mu^s+m)\psi(x)$ it follows that in the $a\to 0$ continuum limit, $S_F[U]$ is gauge invariant and equal to the lattice euclidean fermion action.

Therefore, the (36) path integral can be computed by Wick rotation as

$$\int DAD\psi D\overline{\psi}\exp(-\int\tilde{\mathscr{L}}_E(\psi,\partial\psi,A,\partial A)dt\,d^3\vec{x})O(\psi,\overline{\psi})= \\ =\int DA\exp(-S_G[U])\int D\psi D\overline{\psi}\exp(-S_F[U](\psi,\overline{\psi}))O(\psi,\overline{\psi}) \qquad (37)$$

We can write $-S_F[U](\psi,\overline{\psi})=\overline{\psi}D_W[U]\psi$ where $D_W[U]$ is a matrix acting on the $(\psi(x))_{x\in\Lambda}$ space.

Since ψ , $\overline{\psi}$ can be considered as independent sets of Grassmann variables, with (8) , (8''') relations, we have:

$$Z(\eta,\overline{\eta})=\int D\psi D\overline{\psi}\exp(-S_F[U](\psi,\overline{\psi})+\overline{\eta}\psi+\overline{\psi}\eta)=$$

$=C\det(D_W[U])\exp(-\overline{\eta}D_W^{-1}[U]\eta)$ with C a normalization, discretization dependent constant and so we can compute:

$$\int D\psi D\overline{\psi}\exp(-S_F[U](\psi,\overline{\psi}))O(\psi,\overline{\psi})=$$

$$=C\det(D_W[U])\frac{\partial^{s+n}}{(\prod_{i=1}^{s}\partial\overline{\eta}_{\beta i}^{\alpha i}(x_i))(\prod_{j=1}^{n}\eta_{\delta j}^{\gamma j}(x'_j))}\exp(-\overline{\eta}D_W^{-1}[U]\eta)\Bigg|_{\eta=\overline{\eta}=0}= \qquad (37')$$

$$=C\langle O\rangle_F[U]\left|\det(D_W[U])\right|$$

If,as in (18") the $O(\psi,\overline{\psi})$ requires integration over $(\vec{x}_i)_i$, $(\vec{x}'_j)_j$ we can include that in the $\langle O\rangle_F[U]$ factor.

Therefore the calculation of (36), (37) integral reduces to computation of $C\int D[U]\langle O\rangle_F[U]\exp(-S_G[U])\left|\det(D_W[U])\right|$ where $\int D[U]...$ means integration over the $(A_\mu^b(x))_{x\in\Lambda,b,\mu}$ variables, with C a normalization , discretization dependent constant.

For the free theory, $(U_\mu(x)=\mathbf{I})$ the ∇_μ^s operator becomes ∂_μ^s ,

$\partial_\mu^s f(x)=\dfrac{f(x+a\hat{\mu})-f(x-a\hat{\mu})}{2a}$ and so

$$-S_F[U](\psi,\overline{\psi})=-\int\overline{\psi}(\gamma_\mu\partial_\mu^s+m)\psi d^4x$$

The Fourier transform on the momentum space of $-(\gamma_\mu\partial_\mu^s+m)\psi$ is $-\left(\frac{i}{a}\gamma_\mu\sin(ap_\mu)+m\right)\mathscr{F}\psi$ and the propagator D_W^{-1} satisfies

$-(\gamma_\mu \partial_\mu^s + m) D_W^{-1}(x) = \delta^4(x)$ and so on the momentum space we have

$$\mathscr{F}\, D_W^{-1}(p) = \frac{i a^{-1} \gamma_\mu \sin(a p_\mu) - m}{m^2 + a^{-2}(\sin^2(a p_0) - \sum_j \sin^2(a p_\mu))}$$

The momentum space propagator has a pole at $p^2 = -m^2$ when $a \to 0$ but has more poles, known as doublers , when $\sin^2(a p_0) - \sum_j \sin^2(a p_j) = m^2 a^2$.

Doublers can interact with each other via loop corrections and in computations we remove them by perturbing slightly the $\gamma_\mu \nabla_\mu^s + m$ operator, taking

$$S_F[U](\psi, \overline{\psi}) = -\overline{\psi} D_W[U] \psi = a^4 \sum_{x \in \Lambda} \overline{\psi} \left(\gamma_\mu \nabla_\mu^s + m - \frac{a}{2} \nabla_\mu^{s+} \nabla_\mu^s \right) \psi \ .$$

Monte-Carlo sampling method

Let $P:[0,L]^M \to \mathbb{R}_+$ with P continuous and $\int P(x) d^M x = W < \infty$

Then we have a probability on $[0,L]^M$ given by

$\overline{P}(A) = \int_A \frac{P(x)}{W} d^M x$ for any measurable set in $[0,L]^M$.

For $n = (n_i)_i \in \{0, \dots, q-1\}^M$ we denote $C_n = \prod_{i=1}^M [n_i L/q, (n_i+1) L/q]$

and take a sample $(x_k)_{k=\overline{1,S}}$, $x_k \in [0,L]^M$, $S = q^{M+1}$ such that:

$\mathrm{card}\{k = \overline{1,S} | x_k \in C_n\} = \lfloor S \overline{P}(C_n) \rfloor$

We consider also the measures on $[0,L]^M$ defined by:

$$\varepsilon_k(A) = \begin{cases} 1 & \text{if } x_k \in A \\ 0 & \text{else} \end{cases} , \ \mu_S = \frac{1}{S} \sum_{k=1}^S \varepsilon_k$$

Then for any Borel set A of $[0, L]^M$ with $\overline{P}(\partial A) = 0$ we can show that $(*)$:

$\lim_{q \to \infty} \mu_S(A) = \overline{P}(A)$ and so for any continuous $F:[0,L]^M \to \mathbb{R}$ we have

$$\int F(x) P(x) d^M x = W \int F d\overline{P}(x) = \lim_{q \to \infty} W \int F d\mu_S(x) = \lim_{q \to \infty} \frac{W}{S} \sum_{k=1}^S F(x_k)$$

Now we demonstrate $(*)$:

By compactness of $\overline{A} = A \cup \partial A$, measure definition and density of rational fractions, for large enough $q \in \mathbb{N}$ we find $(nj)_j$, $nj \in \{0, \dots, q-1\}^M$, $nj \neq nl$ for $j \neq l$ such that $\left| \overline{P}(\bigcup_j C_{nj}) - \overline{P}(A) \right| < \varepsilon$,

$$\left| \overline{P}(\bigcup_j C_{nj}) - \mu_S(\bigcup_j C_{nj}) \right| < \frac{q^M}{S} \le \frac{1}{q}$$

$\left| \mu_S(\bigcup_j C_{nj}) - \mu_S(A) \right| < \varepsilon$ with arbitrary positive ε and the result follows.

In a lattice simulation we do the space-time integrations on a bounded hypercube of time interval length *T* and space volume *V* so that we can consider that *Λ* is a finite set of lattice points. We can use Monte-Carlo sampling method to compute the (37) integral.

Let M be the dimension of the $(A_\mu^b(x))_{x\in\Lambda,b,\mu}$ space,

taking $A=(A_\mu^b(x))_{x\in\Lambda,b,\mu}\in[-L/2,L/2]^M$,

$C_n=\prod_{i=1}^{M}[n_i\delta-L/2,(n_i+1)\delta]$ for $n=(n_i)_i\in\{0,...,q-1\}^M$, $\delta=L/q$.

Then we take samples:

$A^{(k)}\in[-L/2,L/2]^M$, $k=\overline{1,S}$, $S=q^{M+1}$, $U_\mu^{(k)}(x)=\boldsymbol{I}-i\sum_g agA_\mu^{(k)b}(x)T^b$

such that for any multi-index *n* we have

card$\{k=\overline{1,S}|A^{(k)}\in C_n\}=\lfloor S\overline{P}(C_n)\rfloor$; $\overline{P}$ is a probability on $[-L/2,L/2]^M$ space

defined by the density $\frac{1}{W}\exp(-S_G[U])|\det(D_W[U])|$

with $W=\int\exp(-S_G[U])|\det(D_W[U])|d^MA$

According to above considerations , the (37) integral can be determined as

$C\lim_{q\to\infty}\frac{W}{S}\sum_{k=1}^{S}\langle O\rangle_F[U^{(k)}]$.

Meson and baryon masses

A scalar meson appears as a combination $\psi_M(x_1,x_2)=\overline{\psi}^a(x_1)\psi^a(x_2)$ with no summation over the colour index $a=\overline{1,3}$ since at a location $\vec{x}$ the quark and antiquark have one colour (anti-colour), taking $x_i=(t,\vec{x}_i)=(t,\vec{x})$ for $i=1,2$.

For the scalar meson we consider an equivalent scalar field of a spin 0 particle having an effective mass m , $\hat{\varphi}=\hat{\varphi}(t,\vec{x})$ as in (4a) and the equivalent propagator from $(0,\vec{x})$ to $(t,\vec{x})$, $t>0$ which is $-i\langle 0|\hat{\varphi}(t,\vec{x})\hat{\varphi}^+(0,\vec{x})|0\rangle$.(38)

Therefore , taking $\hat{F}(t)=\hat{\psi}_M((t,\vec{x}),(t,\vec{x}))$ for a given location $\vec{x}$, the (38) propagator must be similar to Lorentz invariant $-i\langle\hat{F}(t)\hat{F}(0)\rangle$.

After some calculus, according to above established results we derive

$$C(t)=\int D[U]\langle O\rangle_F[U]\exp(-S_G[U])|\det(D_W[U])|=K\int\frac{\exp(-\sqrt{\vec{k}^2+m^2}t)}{\sqrt{\vec{k}^2+m^2}}d^3\vec{k}$$

with K a t , m independent constant and

$O(\psi,\overline{\psi})=(\overline{\psi}^a(t,\vec{x})\psi^a(t,\vec{x}))(\overline{\psi}^a(0,\vec{x})\psi^a(0,\vec{x}))$

A baryon appears as $\psi_B(x_1,x_2,x_3)=(\psi_\alpha^a(x_1)\psi_\beta^b(x_2)\psi_\gamma^c(x_3))_{\alpha\beta\gamma}$, $a\neq b\neq c\neq a$ colour indices, taking $x_i=(t,\vec{x}_i)=(t,\vec{x})$, $i=\overline{1,3}$ at given location $\vec{x}$.

We consider for a spin ½ baryon an equivalent Dirac spinor field having an effective mass m , $\hat{\psi}=\hat{\psi}(t,\vec{x})$ as in (4c) and the equivalent propagator trace from $(0,\vec{x})$ to $(t,\vec{x})$: $-i\langle 0|\hat{\psi}_\alpha(t,\vec{x})\hat{\overline{\psi}}_\alpha(0,\vec{x})|0\rangle$.
Thus similar to above we derive

$$C(t)=\int D[U]\langle O\rangle_F[U]\exp(-S_G[U])\left|\det D_W[U]\right|=Km\int\frac{\exp(-\sqrt{\vec{k}^2+m^2}t)}{\sqrt{\vec{k}^2+m^2}}d^3\vec{k}$$

with K a t,m independent constant and the Lorentz invariant

$$O(\psi,\overline{\psi})=\psi_\alpha^a(t,\vec{x})\,\psi_\beta^b(t,\vec{x})\,\psi_\gamma^c(t,\vec{x})\,\overline{\psi}_\alpha^a(0,\vec{x})\,\overline{\psi}_\beta^b(0,\vec{x})\,\overline{\psi}_\gamma^c(0,\vec{x})$$

Focussing on the baryon case, integrating in spherical coordinates and then by parts we obtain, after a variable changing:

$C(t)=\dfrac{4\pi Km}{t^2}\displaystyle\int_0^\infty\exp(-\sqrt{k^2+m^2t^2})dk$ and so for $G(t)=tC(t)$ we have

$G(t)=4\pi Km^2P(mt)$ where $P(z)=\dfrac{1}{z}\displaystyle\int_0^\infty\exp(-\sqrt{k^2+z^2})$, $z=mt$,

$G'(t)=4\pi Km^3P'(z)$, $\dfrac{G'(t)}{G(t)}=m\dfrac{P'(z)}{P(z)}=-m\left(\dfrac{1}{z}+H(z)\right)$ with

$$H(z)=\left(\int_0^\infty\frac{z\exp(-\sqrt{k^2+z^2})}{\sqrt{k^2+z^2}}dk\right)\left(\int_0^\infty\exp(-\sqrt{k^2+z^2})dk\right)^{-1}=$$

$=\left(\displaystyle\int_0^\infty\frac{u\exp(-\sqrt{1+k^2}/u)}{\sqrt{1+k^2}}dk\right)\left(\int_0^\infty u\exp(-\sqrt{1+k^2}/u)dk\right)^{-1}$ with $z=\dfrac{1}{u}$

Variable changing to $s=\exp(-\sqrt{1+k^2}/u)$ leads to

$$\int_0^\infty u\exp(-\sqrt{1+k^2}/u)dk=\int_1^\infty\exp(-\tau/u)\,\tau/\sqrt{\tau^2-1}\,d\tau=$$

$$=\int_1^\infty u\sqrt{\tau^2-1}\exp(-\tau/u)d\tau=\int_0^h u\sqrt{u^2\ln^2(s)-1}\,ds$$

$\displaystyle\int_0^\infty\frac{u\exp(-\sqrt{1+k^2}/u)}{\sqrt{1+k^2}}dk=\int_0^h\frac{u^2}{\sqrt{u^2\ln^2(s)-1}}ds$ where $h=\exp(-z)$.

Hence after some calculus we obtain:

$$H(z)=\left(\int_0^1\frac{|\ln(h)|^{1/2}}{(\ln^2(s)+2\ln(s)\ln(h))^{1/2}}ds\right)\left(\int_0^1\frac{(\ln^2(s)+2\ln(s)\ln(h))^{1/2}}{|\ln(h)|^{1/2}}ds\right)^{-1} .$$

We can verify that for $s\in(0,1)$ we have:

$$\frac{|\ln(h)|^{1/2}}{(\ln^2(s)+2\ln(s)\ln(h))^{1/2}}<\frac{1}{\sqrt{2|\ln(s)|}} ,$$

$\displaystyle\int_0^1\frac{1}{\sqrt{|\ln(s)|}}ds=\int_0^\infty\tau^{-1/2}\exp(-\tau)d\tau=\Gamma(1/2)$ and for $z>1$ also

$$\frac{(\ln^2(s)+2\ln(s)\ln(h))^{1/2}}{|\ln(h)|^{1/2}}<(1+\sqrt{2})|\ln(s)| \ ,$$

$\int_0^1 |\ln(s)|ds=\int_0^\infty \tau\exp(-\tau)d\tau<\infty$ and so by dominated convergenge for $z\to\infty$,
it follows that

$$\lim_{z\to\infty} H(z)=\left(\int_0^1 (2|\ln(s)|)^{-1/2}ds\right)\left(\int_0^1 (2|\ln(s)|)^{1/2}ds\right)^{-1}=$$

$$=\frac{1}{2}\left(\int_0^\infty s^{-1/2}\exp(-s)ds\right)\left(\int_0^\infty s^{1/2}\exp(-s)ds\right)^{-1}=(1/2)\Gamma(1/2)(\Gamma(3/2))^{-1}=1 \ .$$

Therefore $\lim_{t\to\infty} -\frac{G'(t)}{G(t)}=m$ and so for large t we can consider that

$$\frac{\ln C(t)-\ln C(t+a)}{a}=m \ , \ ma=\ln\left(\frac{C(t)}{C(t+a)}\right) \qquad (39)$$

In the same way , the (39) relation results valid also for the meson case.
Notice that in the ψ_M , ψ_B expressions we have supressed flavour differences between the various ψ factors, so that we can have mesons made from an up-quark and a down-antiquark for example or baryons made from two up-quarks and one down-quark like the proton for example. Also we make the location variables $\vec{x}_i$ equal to the same $\vec{x}$ only after computing $\langle O\rangle_F[U]$ according to (37') Wick contraction relation, since otherwise, because we consider the quark fields variables as Grassmann variables in the integration, we would have a vanishing $O(\psi,\overline{\psi})$ operator value due to appearing of squared Grassmann variables in the expression of $O(\psi,\overline{\psi})$. The locations of the quarks / antiquarks in a many-quark system as a meson or a baryon can be considered to be approximatively the same (due to quark confinement), but however not identically the same.

Consider the SU(3) quantum chromodynamics theory with two degenerate quark flavours , the up-quark and the down-quark with equal masses $m = m_u = m_d$. The boson masses are vanishing , $M_a = 0$.
The lattice action depends on two free parameters:
- the quark mass m;
- the value of the strong interaction coupling *g* (which can be absorbed into the *A* integration variable).

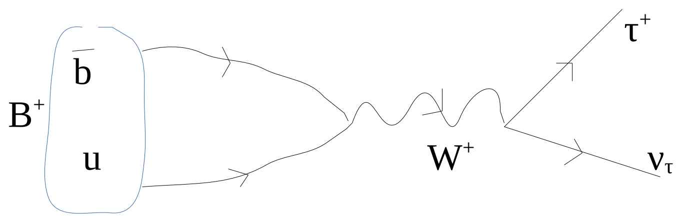
Then we can compute (in dependence of m) , for a lattice spacing *a* the masses of the π meson (made of an up-quark and a down-antiquark) and the proton p (made of two up-quarks and a down-quark): only the dimensionless quantities am_π and am_p can be computed, according to (39).
From experiments we can determine the fraction $(m_\pi/m_p)_{exp}$ and so we can tune the quark mass m such that the lattice simulation computed $(m_\pi/m_p)_{lat}$ matches the experimental $(m_\pi/m_p)_{exp}$. Then we determine the spacing *a* in physical units from $(am_\pi)_{lat}$ and m_π^{phys} .

The continuum limit must be taken using the constant line of physics m_π , $m_p << a^{-1}$ while keeping $(m_\pi/m_p)_{lat}$ constant.

With relations (39) we are able to compute effective masses of mesons , baryons and even atomic nuclei which are made of nucleons which are protons (two up- and one down- quark) and neutrons (two down- and one up- quark) and can be considered as a system of many quarks hold together by the strong interaction. The most part of their particles masses are then given by the gluon intermediated interaction energy. The strong interaction, intermediated by the SU(3) gluons (in the unified SU(3)xSU(2)xU(1) theory) has a positive contribution to the nucleon binding energy in an atomic nucleus while the weak and electromagnetic interaction , intermediated by the SU(2)xU(1) gluons which for positive electric charged protons turns out to be a repelling (Coulombian) force has a negative contribution to the nucleon binding energy. Thus for large (heavy) atomic nuclei the negative binding energy (as an absolute value) can exceed the positive binding energy, because the weak and electromagnetic interaction becomes more significant as the dimension of the nucleus increases. Therefore the fusion of two light nuclei to another light nucleus happens with energy emission and the fission of a heavy atomic nucleus happens also with an emission of energy. The energy gain per fission event ΔE can be computed as $\Delta E = \Delta mc^2$,where Δm is the difference between the sum of effective masses of the outgoing from the fission particles (atomic nuclei and other hadrons) and the sum of incoming in the fission interaction particles (atomic nucleus to be fissioned and the fission event producing particle which can be for example a neutron) effective masses.

To allow transitions between different flavours of quarks for decays like

$$B^+ = \bar{b}u \xrightarrow{W^+} \tau^+ + \bar{\nu}_\tau \quad (40)$$



where the notations are:
b – for the bottom-quark , u – for the up-quark,
W^+ for the combination of $W^1 + i\,W^2$ first two weak SU(2) bosons,
τ^+ - for the tau-antimuon , ν_τ - for the tau-neutrino,
B^+ - for the b – meson in which a bottom-antiquark and an up-quark are confined by the strong interaction,
we add to the SU(3)xSU(2)xU(1) theory Lagrangian density, weak interaction terms like

$g\,\overline{\psi}^{b\alpha}\gamma^{\mu}\left(\frac{\mathbf{I}-\gamma^5}{2}\right)W^{+}_{\mu}\,\psi^{u\alpha}$, with b,u flavours , α colour index and $\gamma^5=i\,\gamma^0\gamma^1\gamma^2\gamma^3$ or equivalently we add $-i\,g\,\overline{\psi}^{b\alpha}\gamma_{\mu}\left(\frac{\mathbf{I}-\gamma^5}{2}\right)W^{+}_{\mu}\,\psi^{u\alpha}$ to the euclidean Lagrangian density.

Obviously in the lattice simulation we have $W^{+}_{\mu}=(2i/(a\,g))\operatorname{tr}(U_{\mu}-\mathbf{I})(T^1+i\,T^2)$ where $2T^1$ and $2T^2$ correspond to the σ_1 respective σ_2 Pauli matrices , from the generators of SU(2) and *g* is the weak coupling constant.

Since it is a weak coupling we can have a perturbative approach and for the decay transition (40) we have to compute expressions for an operator

$$O(\psi,\overline{\psi},U)=\int\Big(\overline{\psi}_i^{\tau}(x_1)\,\psi_j^{\nu}(x_2)(-i\,g)\,\overline{\psi}^{b\alpha}(x)\,\gamma_{\mu}\left(\frac{\mathbf{I}-\gamma^5}{2}\right)W^{+}_{\mu}(x)$$

$$\psi^{u\alpha}(x)\,\psi_k^{b\alpha}(y_1)\,\overline{\psi}_l^{u\alpha}(y_2)\Big)d^4x$$

and with $x_s=(T,\vec{x}_s)$, $y_s=(0,\vec{y}_s)$, $s=1,2$ we take for $i,j,k,l=\overline{0,3}$:

$$A_{lat}^{ijkl}(\vec{x}_1,\vec{x}_2,\vec{y}_1,\vec{y}_2)=\int D[U]\langle O\rangle_F[U]\exp(-S_G[U])\left|\det D_W[U]\right| \quad (41)$$

On the other hand, in the electroweak interaction theory, inter-flavour transitions can be allowed by considering mixed down-type weak interaction partners (d’ , s’ , b’) to the (u , c , t) up-type quarks given by unitary Cabibo-Kobayashi-Maskawa matrix

$$V_{CKM}=\begin{pmatrix} V_{ud} & V_{us} & V_{ub}\\ V_{cd} & V_{cs} & V_{cb}\\ V_{td} & V_{ts} & V_{tb}\end{pmatrix} \quad \text{with} \quad \begin{pmatrix} d'\\ s'\\ b'\end{pmatrix}=V_{CKM}\begin{pmatrix} d\\ s\\ b\end{pmatrix}$$

$|V_{ij}|^2$ is the transition probability from a flavour j quark to a flavour i quark so that the significant changed part of the electroweak Lagrangian density will be

$$\mathscr{L}_{W^{\pm}}=g(\overline{u}\ \ \overline{c}\ \ \overline{t})\,\gamma^{\mu}\left(\frac{\mathbf{I}-\gamma^5}{2}\right)V_{CKM}\begin{pmatrix} d\\ s\\ b\end{pmatrix}W^{-}_{\mu}+g(\overline{d}\ \ \overline{s}\ \ \overline{b})\,V^{+}_{CKM}\gamma^{\mu}\left(\frac{\mathbf{I}-\gamma^5}{2}\right)\begin{pmatrix} u\\ c\\ t\end{pmatrix}W^{+}_{\mu}$$

(The $\frac{\mathbf{I}-\gamma^5}{2}$ appearing since only the left-handed fields participate in the weak interaction.

An equivalent to A_{lat}^{ijkl} in the modified electroweak theory according to $\mathscr{L}_{W^{\pm}}$ is

$$A_W^{ijkl}(\vec{x}_1,\vec{x}_2,\vec{y}_1,\vec{y}_2)=V^{*}_{ub}\int\Big\langle 0\Big|\widehat{\overline{\psi}}_i^{\tau}(x_1)\,\widehat{\psi}_j^{\nu}(x_2)\,\widehat{\overline{\psi}}^{\nu}(y)\,\gamma^{\mu}\left(\frac{\mathbf{I}-\gamma^5}{2}\right)\widehat{\psi}^{\tau}(y)$$

$$\widehat{W}^{-}_{\mu}(y)\,\widehat{W}^{+}_{\lambda}(y')\,\widehat{\overline{\psi}}^{b\alpha}(y')\,\gamma^{\lambda}\left(\frac{\mathbf{I}-\gamma^5}{2}\right)\widehat{\psi}^{u\alpha}(y')\,\widehat{\psi}_k^{b\alpha}(y_1)\,\widehat{\overline{\psi}}_l^{u\alpha}(y_2)\Big|0\Big\rangle d^4y\,d^4y'=$$

$$=C V_{ub}^{*} \int \left((D^{\nu}(x_2 - y)\, \gamma^{\mu} (\mathbf{I} - \gamma^5) D^{\tau}(y - x_1))_{ji} D^{bos}_{\mu\lambda}(y - y') \right. \qquad (42)$$
$$\left. (D^{b}(y_1 - y')\, \gamma^{\lambda} (\mathbf{I} - \gamma^5) D^{u}(y' - y_2))_{kl} \right) d^4 y\, d^4 y'$$

where C is a constant.

Corresponding to the A^{ijkl} we have the momentum dependent amplitude:

$$B(p,q,s,h) = \int \left(\exp(ipx_2) \bar{u}_j^{\nu}(p) (E_p/m_{\nu})^{1/2} \exp(iqx_1) v_i^{\tau}(q) (E_q/m_{\tau})^{1/2} \right.$$
$$\exp(-isy_1) \bar{v}_k^{b}(s) (E_s/m_b)^{1/2} \exp(-ihy_2) u_l^{u}(h) (E_h/m_u)^{1/2}$$
$$\left. A^{ijkl}(\vec{x}_1, \vec{x}_2, \vec{y}_1, \vec{y}_2) \right) d^3 \vec{x}_1 d^3 \vec{x}_2 d^3 \vec{y}_1 d^3 \vec{y}_2$$

After some calculations, considering (42) and (13) , (14) type relations we can derive:

$$B_W(p,q,s,h) = C V_{ub}^{*} \left(\frac{m_{\nu} m_{\tau} m_b m_u}{E_p E_q E_s E_h} \right)^{1/2} (\bar{v}^{b}(s)\, \gamma^{\lambda} (\mathbf{I} - \gamma^5) u^{u}(h))$$
$$(\bar{u}^{\nu}(p)\, \gamma^{\mu} (\mathbf{I} - \gamma^5) v^{\tau}(q)) \left(-\eta^{\mu\lambda} + \frac{(p+q)_{\lambda} (p+q)_{\mu}}{M^2} \right) \frac{1}{(p+q)^2 - M^2}$$

where C is a constant which can depend on the interaction time interval T and the interaction space volume, since we consider momentum conservation and incoming and outgoing momenta on mass shell , having therefore

$p+q=s+h$, $\delta^4(p+q-s-h) = \frac{VT}{(2\pi)^4}$ and M is the W-boson mass.

Notice that for a given quark or lepton and given four-momentum p on mass shell, the $u(p)$, $v(p)$ Dirac spinors are defined by their normalization values in the rest frame , where $\vec{p}=0$ and spin index variable 1 ,2 is supposed to be understood.

Therefore we have a constant *C'* depending on *V*, *T*, *g* and discretization and normalization of Grassmann variables of the lattice simulation, such that

$B_W(p,q,s,h) = C' B_{lat}(p,q,s,h)$ where we take $p+q=s+h$ and the four-momenta p,q,s,h are on mass shell. (43)

From (43) we can extract in some momentum range a value $C V_{ub}^{*}$ where C is a lattice simulation dependent constant and similarly $C V_{ud}^{*}$ and $C V_{us}^{*}$ with the same constant.

Requiring that V_{CKM} is an unitary matrix and so $|V_{ub}|^2 + |V_{ud}|^2 + |V_{us}|^2 = 1$,

we obtain the values of V_{ub} , V_{us} , V_{ud} and in the same way the whole V_{CKM} to multiplication with global phase factors which can be absorbed into the quark field functions.

As we know, SU(N) requires a basis of $N^2 - 1$ hermitean traceless matrices as generators, and so a matrix $U \in SU(N)$ requires $N^2 - 1$ real parameters. Adding one real parameter to determine the determinant of absolute value 1, we obtain that an unitary CKM NxN matrix requires N^2 real parameters. 2 N – 1 of these parameters are not physically significant because one phase factor can be absorbed into each quark field (both of the mass eigenstates and the weak primed eigenstates of the N

down-type flavours) but the matrix is independent of a common phase. Hence the total number of free variables independent of the choice of the phases of basis vectors is N^2 - (2 N – 1) = $(N – 1)^2$.
Splitting suitable chosen generators of SU(N), which are complex hermitian traceless matrices into real and pure imaginary generators we show without difficulties that an unitary matrix V can be expressed as $V = \exp(A + iB)$ where A is a real antisymmetric matrix having all diagonal elements equal to zero and *B* is a real symmetric matrix. Therefore from the $(N – 1)^2$ free real variables which remained to define the CKM matrix, N (N – 1) / 2 are rotation angles (the *A* matrix above) which are the so called quark mixing angles .
The remaining (N – 1) (N – 2) /2 are imaginary phase variables which cause CP-violation as we will show.
For N = 2 we have no complex phase factors an one quark mixing angle. For N = 3 there are three mixing angles and one CP- violating complex phase. For CP- violation to occur we must have at least three families of quarks.
To create an imbalance of matter and antimatter, for the Universe to exist, from an initial condition of balance, a necessary condition is the existence of CP- violation, or equivalent, considering the CPT theorem, the existence of time reversal T- violation, so at least three families of quarks exist in nature.
The reason why a complex phase factor in $(V_{ij})_{i,j}$ causes CP- violation can be seen as follows:
Consider any given particles (or sets of particles) a and b and their antiparticles $\bar{a}$ and $\bar{b}$. Now consider the processes $a \to b$ and the corresponding antiparticle processes $\bar{a} \to \bar{b}$ under CP transformation,denote their amplitudes M respectively $\bar{M}$. Before CP- violation, these terms must be the same complex number $M = \bar{M}$.
Let $M = |M| \exp(i\theta)$. If a phase factor is introduced (from the CKM matrix), denote it $\exp(i\varphi)$.
$\bar{M}$ contains the conjugate matrix to M , so it picks up a phase factor $\exp(-i\varphi)$.
Now we have : $M = |M| \exp(i\theta) \exp(i\varphi)$, $\bar{M} = |M| \exp(i\theta) \exp(-i\varphi)$.
Physically measurable reaction rates are proportional to $|M|^2 = |\bar{M}|^2$.
However, consider that are two different routes $a \overset{1}{\to} b$ and $a \overset{2}{\to} b$, or equivalently two unrelated intermediate states $a \to 1 \to b$ and $a \to 2 \to b$ and we have:

$$M = |M_1| \exp(i\theta_1) \exp(i\varphi_1) + |M_2| \exp(i\theta_2) \exp(i\varphi_2)$$

$$\bar{M} = |M_1| \exp(i\theta_1) \exp(-i\varphi_1) + |M_2| \exp(i\theta_2) \exp(-i\varphi_2) \text{ and so}$$

$$|M|^2 - |\bar{M}|^2 = -4 |M_1| |M_2| \sin(\theta_1 - \theta_2) \sin(\varphi_1 - \varphi_2).$$

Thus we see that a complex phase factor gives rise to processes that proceed at different rates for particles and antiparticles and CP is violated.

There can be considered also a lepton mixing matrix or neutrino mixing matrix, which contains information on the mismatch of quantum states of the three flavours of neutrinos ν_e , ν_τ , ν_μ in the charged current weak interaction with the lepton partners e , τ , μ . That matrix is an unitary matrix, called the

Pontecorvo-Maki-Nakagawa-Sakata matrix, PMNS.

Random walk, mean free path and critical mass of a fissile material

A random walk is a random process that describes a path that consists of a succession of random steps on some mathematical space.
A lattice random walk is a random walk on a regular lattice where at each step the location jumps to another site according to some probability distribution.
In a simple symmetric random walk the location can jump only to neighbouring sites of the lattice forming a lattice path and the probabilities of the location jumping to each one of its immediate neighbours are the same.
Consider a tridimensional lattice $\Lambda=\{(n_i a)_{i=\overline{1,3}} | n_i \in \mathbb{Z}\ ,\ i=\overline{1,3}\}$.
To define the random walk we consider the product probability space of succession of steps:

$S=(\prod_{i\in\mathbb{N}^*}\{-1,1\}^3\ ,\ \hat{P}=\bigotimes_{i\in\mathbb{N}^*}(\bigotimes_1^3 P))$ with $P(\{-1\})=P(\{1\})=\frac{1}{2}$ and the independent random variables $Z_{i\alpha}:S\to\{-a,a\}$ with $Z_{i\alpha}((x_j^\beta)_{j\in\mathbb{N}^*\ ,\ \beta=\overline{1,3}})=x_i^\alpha a$.

We have $E(Z_{i\alpha})=\int Z_{i\alpha}d\hat{P}=0$ and we take $\vec{Z}_i=(Z_{i\alpha})_{\alpha=\overline{1,3}}\ ,\ \vec{S}_n=\sum_{i=1}^n \vec{Z}_i$.

In order for $S_{n\alpha}$ to be $k_\alpha a$ it is necessary and sufficient that the number of +1 steps in α direction excceds the number of -1 steps taken in α direction of the n steps defined tridimensional walk. Therefore, for the α direction, $+1$ step must be taken $(n+k_\alpha)/2$ times from a total of n steps . The total number of n steps considered tridimensional walks is 2^{3n}. Therefore we can derive

$$\hat{P}\circ\vec{S}_n^{-1}(\{(k_1 a,k_2 a,k_3 a)\})=\prod_{\alpha=1}^3\left(\binom{n}{(n+k_\alpha)/2}\frac{1}{2^n}\right)$$ which implies $n\equiv k_\alpha \pmod 2$

for the probability not be equal to 0.
Using the Stirling formula : $\lim_{n\to\infty}\frac{\sqrt{2\pi n}(n/\mathrm{e})^n}{n!}=1$ after some calculus we obtain

$$\ln\left(\binom{n}{(n+k_\alpha)/2}\frac{1}{2^n}\right)\approx\frac{k_\alpha^2}{2n}-\frac{1}{2}\ln n+\ln\sqrt{\frac{2}{\pi}}$$ for large n.

Therefore the asymptotic probability distribution for the defined tridimensional random walk as the number of steps increases when the step length is constant for each step is a function of the radius from the origin $\rho=\rho(r)$ having

$$\hat{P}\circ\vec{S}_n^{-1}(A)\simeq\int_A\rho(r)\,dr\,d\Omega=\int_A\left(\frac{2}{n\pi}\right)^{3/2}r^2\exp\left(\frac{r^2}{2n}\right)dr\,d\Omega$$

$d\Omega$ - solid angle , $\rho(r)=\left(\frac{2}{n\pi}\right)^{3/2}r^2\exp\left(\frac{r^2}{2n}\right)$

Also we can compute :

$$E(|S_{2n\alpha}|)=\int |S_{2n\alpha}| d\hat{P}=a\sum_{k=0}^{n} 2k\binom{2n}{n+k}\frac{1}{2^{2n}}=$$

$$=\frac{a}{2^{2n}}\sum_{k=0}^{n}\left((n+k)\binom{2n}{n+k}-(n-k)\binom{2n}{n-k}\right)=$$

$$=\frac{a}{2^{2n}}\left(n\binom{2n}{n}+\sum_{k=0}^{2n} k\binom{2n}{k}-2\sum_{k=0}^{n} k\binom{2n}{k}\right)$$

We have:

$$k\binom{2n}{k}=2n\binom{2n-1}{k-1}$$

$$\sum_{k=1}^{n}\binom{2n-1}{k-1}=\sum_{k=0}^{n-1}\binom{2n-1}{k}=\sum_{k=n}^{2n-1}\binom{2n-1}{k}=\sum_{k=n+1}^{2n}\binom{2n-1}{k-1}$$

and therefore we obtain

$E(|S_{2n\alpha}|)=\frac{a}{2^{2n}}n\binom{2n}{n}$ and using the Stirling formula it follows

$E(|S_{2n\alpha}|)\simeq\frac{a}{\sqrt{2\pi}}\sqrt{2n}$ for large $n.$ The net distance travelled in a lattice simple random walk is proportional to the square root of the number of steps.

The mean free path is the average distance over which a moving particle (such as an atom, molecule , photon or neutron), travels before substantially changing its direction or energy, typically as a result of one or more successive collisions with other particles.

Imagine a beam o particles being shot through a target and consider an infinitesimally thin slab of the target. The area of the slab is L^2 (L is the width and height of the slab) and its volume is L^2 dx (dx is the thickness of the infinitesimal slab). The concentration of the atoms in the slab is n. The typical number of stopping atoms in the slab is then n L^2 dx . If *l* is the mean free path, then the probability of stopping within the distance *l* must be equal to 1: $\wp(\text{stopping within } l)=1$.

The probability that a beam particle will be stopped in the slab of thickness dx is the net area of the stopping atoms (which is the scattering cross section times the number of stopping atoms in the slab) divided by the total area of the slab:

$$\wp(\text{stopping within } dx)=\frac{\sigma n L^2 dx}{L^2}=n\sigma dx \quad .$$

Hence the mean free path is $l=(n\sigma)^{-1}$ where σ is the scattering cross section.

Consider now a fissile material of atoms in which fission events are produced by an existent neutron population. A neutron can scatter on atoms of the material, changing its momentum, or produce a fission event on an atom releasing other neutrons which can cause further fission events, leading to a chain reaction. If the effective neutron multiplication factor *k*, the average number of neutrons released per fission event that go on to cause another fission event rather than being absorbed or leaving the material, is equal to 1 (k = 1) the mass is critical and the chain reaction is self sustaining.

Most interactions of neutrons with the material are scattering events, so that a given neutron obeys a random walk until it either escapes from the medium or causes a fission reaction. If k = 1 we can consider that we have the same neutron travelling a random walk of $n_s + n_f$ steps experiencing n_s scattering events and n_f fission events and during the fission event steps the neutron travels a net distance corresponding to a mean scattering free path l, while the total net distance travelled during both fission event steps and scattering event steps together will be R_c, the radius of a spherical critical mass.

Since the number of steps squared is proportional to the distance travelled in a random walk we have: $\frac{R_c}{l}=\sqrt{s}$ with $s=1+\frac{n_s}{n_f}$.

Also, if σ is the neutrons on atoms scattering process effective cross section and n is the nuclear number density of atoms we have $l=(\sigma n)^{-1}$ and so $R_c \simeq \frac{\sqrt{s}}{\sigma n}$. If M is the critical mass, ρ is the density of the material and m is the mass of one atom of the material, we will have:

$M=\rho\frac{4}{3}\pi R_c^3$, $n=\frac{3}{4\pi}\frac{M}{m}\frac{1}{R_c^3}$, $R_c \simeq \frac{\sqrt{s}}{\sigma}\frac{m}{M}R_c^3\frac{3}{4\pi}$ and generally

$1 \simeq \frac{f\sigma}{m\sqrt{s}}\rho^{2/3}M^{1/3}$ where f is a factor which takes into account geometrical and other effects. The critical mass depends inversely on the square of density.

In a theory with neutrons and atomic kernels as confined quarks, we should be able to compute in lattice gauge simulation, according to (23) relations the differential cross section for scattering of neutrons on atomic kernels $d\sigma$ and the differential cross section for the fission event process $d\sigma_f$ taking $|\vec{v}_1-\vec{v}_2|=|\vec{v}|=v$ as the absolute value of the neutrons velocity by its thermodynamic average

$v=\sqrt{\frac{2\langle\varepsilon\rangle}{m_0}}=\sqrt{\frac{3k_bT}{m_0}}$ with k_b Boltzmann constant ,

T temperature, m_0 neutron mass.

Then we can determine $\sigma=\int d\sigma$, $\sigma_f=\int d\sigma_f$, $s=1+\frac{\sigma}{\sigma_f}$.

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