Representations of the restricted Lorentz group

Consider the restricted Lorentz group $G=SO^+(3,1)$. We have the Lie group stucture on G defined by zhe mappings:
$\mathbb{R}^6 \ni (\vec{\theta},\vec{\chi}) \rightarrow \exp(\vec{\theta}\bar{\vec{J}}+\vec{\chi}\bar{\vec{K}}) \in SO^+(3,1)$ where $\bar{\vec{J}}=(\bar{J}_k)_{k=\overline{1,3}}$, $\bar{\vec{K}}=(\bar{K}_k)_{k=\overline{1,3}}$ are the generators of G (see Chap. Spin representations and Chap. Rotations and restricted Lorentz groups).
We have the commutation relations:
$[\bar{J}_i,\bar{J}_j]=\epsilon_{ijk}\bar{J}_k$, $[\bar{K}_i,\bar{K}_j]=-\epsilon_{ijk}\bar{J}_k$, $[\bar{J}_i,\bar{K}_j]=\epsilon_{ijk}\bar{K}_k$ $\quad (1)$ and taking

$\bar{M}_{\pm l}=\frac{1}{2}(i\bar{J}_l \mp \bar{K}_l)$ for $l=\overline{1,3}$ we have $[\bar{M}_{+l},\bar{M}_{-k}]=0$ and $[\bar{M}_{\pm l},\bar{M}_{\pm k}]=i\,\epsilon_{lkj}\bar{M}_{\pm j}$ for $l,k=\overline{1,3}$. Let $\bar{X}_{\pm}=\bar{M}_{\pm 1}+i\bar{M}_{\pm 2}$, $\bar{Y}_{\pm}=\bar{M}_{\pm 1}-i\bar{M}_{\pm 2}$, $\bar{H}_{\pm}=2\bar{M}_{\pm 3}$ and we have

$$\exp(\vec{\theta}\bar{\vec{J}}+\vec{\chi}\bar{\vec{K}})=\exp((-i\vec{\theta}-\vec{\chi})\bar{\vec{M}}_{+}+(-i\vec{\theta}+\vec{\chi})\bar{\vec{M}}_{-})=$$
$$=\exp(-i(\vec{\theta}-i\vec{\chi})\bar{\vec{M}}_{+})\exp(-i(\vec{\theta}+i\vec{\chi})\bar{\vec{M}}_{-})$$ because $\bar{M}_{+l}$ and $\bar{M}_{-k}$ commute.

For W a finite dimensional vector space, as a representation U of G (see for definition Chap. Spin representations), we can consider the corresponding generators of the representation:
$H_{\pm},X_{\pm},Y_{\pm}\in GL(W)\simeq M_{n\times n}(\mathbb{C})$ with $n=\dim W$ and we have

$[S_-,S_+]=0$ for any $S_-\in\{H_-,X_-,Y_-\}$, $S_+\in\{H_+,X_+,Y_+\}$,

$$U(\exp(\vec{\theta}\vec{J}+\vec{\chi}\vec{K}))=$$
$$=\exp(-\frac{1}{2}i(\theta_1-i\chi_1)(X_++Y_+)-\frac{1}{2}i(-i\theta_2-\chi_2)(X_+-Y_+)-\frac{1}{2}i(\theta_3-i\chi_3)H_+)\cdot$$
$$\cdot\exp(-\frac{1}{2}i(\theta_1+i\chi_1)(X_-+Y_-)-\frac{1}{2}i(-i\theta_2+\chi_2)(X_--Y_-)-\frac{1}{2}i(\theta_3+i\chi_3)H_-).$$

For $(H,X,Y)\in\{(H_{\pm},X_{\pm},Y_{\pm})\}$ it follows
$[X,Y]=H$, $[H,X]=2X$, $[H,Y]=-2Y$ and that for any eigenvector $v\in W$ of H , $Hv=\lambda v$, $\lambda\in\mathbb{C}$, $v\neq 0$ we will have: $HX^jv=(\lambda+2j)X^jv$ for any $j\in\mathbb{N}$.
The space W being finite dimensional we take
$i_0=\max\{i\in\mathbb{N}|X^iv\neq 0\}$. Let $v_0=X^{i0}v$, $v_j=Y^jv_0$ and it follows
$Hv_j=(\lambda+2(i_0-j))v_j$ so that taking $m=\max\{i\in\mathbb{N}|v_i\neq 0\}$ for $j\in\mathbb{N}$ we have:

$$Xv_0=0\ ,\ Xv_{j+1}=YXv_j+Hv_j\ ,\ Yv_j=v_{j+1}\ ,\ Yv_m=0\,.$$

v_0 , v_1 , … , v_m are linearly independent, being eigenvectors of H for distinct eigenvalues, and by induction follows that H , X , Y satisfy

$S(V)\subseteq V$ for $S\in\{H,X,Y\}$, $V=\mathrm{Sp}[v_0,\dots,v_m]$.

Since $H v_j=(\lambda+2(i_0-j))v_j$ we have $\mathrm{tr}\,H\Big|_V=\sum_{j=0}^{m}(\lambda+2(i_0-j))=\mathrm{tr}[X,Y]\Big|_V=0$.
Therefore $\lambda=m-2i_0\in\mathbb{Z}$, $H v_j=(m-2j)v_j$, $Y v_m=0$, $X v_0=0$, $Y v_j=v_{j+1}$ for $j=\overline{0\ ,\ m-1}$. By induction we also prove that $X v_j=j(m-j+1)$ for $j=\overline{1,m}$

We notice that $V=\mathrm{Sp}[v_0,\dots,v_m]$ is an irreducible representation space, that is, if $V'\subseteq V$ such that $H(V')\subseteq V'$, $X(V')\subseteq V'$, $Y(V')\subseteq V'$ then $V'\in\{\{0\}\ ,\ V\}$.
A representation W of $g=\mathrm{Sp}[\bar{H},\bar{X},\bar{Y}]$ is given by $H,X,Y\in GL(W)$ with $[H,X]=2X$, $[H,Y]=-2Y$, $[X,Y]=H$ and any irreducible representation of g is a $V=\mathrm{Sp}[v_0,\dots,v_m]$ type representation as presented above.
We will prove now that if W is a finite dimensional representation of g then H is diagonalizable and following the introductory considerations we have

$W=\bigoplus_{i=1}^{n}V_i$ with V_i irreducible representations $V_i=\mathrm{Sp}[v_{i0},\dots v_{imi}]$ as above .
Indeed suppose we proved that H is diagonalizable and we have set $V_i=\mathrm{Sp}[v_0^{(i)},\dots,v_{mi}^{(i)}]$, $W'=\bigoplus_{i=1}^{n}V_i$ with V_i irreducible representation for $i=\overline{1,n}$.
If $W'\neq W$, since H is diagonalizable we choose $v\in W\setminus W'$, $\lambda\in\mathbb{C}$, $H v=\lambda v$.
Let $V=\mathrm{Sp}[v_0,\dots,v_m]$ the irreducible representation to which $v=v_k$ belongs.
Assuming that $v_l\in W'$ we have $v_l=\sum_{k,i}\alpha_{ki}Y^k v_0^{(i)}$ and applying X^l we obtain

$$v_0=\sum_{p,q}\beta_{pq}Y^p v_0^{(q)}$$ and further applying Y^k we obtain $v_k=\sum_{s,j}\gamma_{sj}Y^s v_0^{(j)}$

contradicting $v\in W\setminus W'$. Thus we can conclude $W=\bigoplus_{i=1}^{n}V_i$.
Let the induction over dim*W* assumption be: “For any representation *W* of *g* with dim*W* = *k* < *n* the corresponding *H* generator is diagonalizable.”
Let *W* with dim*W* = *n* the invariant representation vector space for a representation of *g*. Assuming that the corresponding *H* generator is not diagonalizable, from its Jordan canonical form, we derive

$W=W_0\oplus W_1$, $W_1=\mathrm{Sp}[v_i]_{i=\overline{1,m}}$, $(H-\lambda_i\boldsymbol{I})v_i=0$, $v_i\neq 0$, $\lambda_i\in\mathbb{C}$ for $i=\overline{1,m}$
$(H-\lambda_i\boldsymbol{I})w_{ij}=w_{i\,j-1}$, $j=\overline{1,m_i}$, $w_{i0}=v_i$ and at least one $i\in\{1,\dots,m\}$ we have $m_i\geq 1$
$W_0=\mathrm{Sp}[w_{ij}]_{\substack{i=\overline{1,m}\\ j=\overline{1,mi}}}$

Since the commutation relations lead to

$(H-\lambda\boldsymbol{I})X=X(H-(\lambda-2)\boldsymbol{I})$, $(H-\lambda\boldsymbol{I})Y=Y(H-(\lambda+2)\boldsymbol{I})$ for any $\lambda\in\mathbb{C}$,
we can derive that for $W'=\mathrm{Sp}[\{w\in W|$ exists $\lambda\in\mathbb{C}$ such that $(H-\lambda\boldsymbol{I})^2 w=0\}]$ (2)
we have $H(W')\subseteq W'$, $X(W')\subseteq W'$, $Y(W')\subseteq W'$.

If $W'\neq W$, obviously $\dim W'<n$ and by induction hypothesis H is diagonalizable on W' which contradicts H not diagonalizable.
Therefore we can assume that $W'=W$ and so $m_i\leq 2$ for $i=\overline{1,m}$ and we have
$k,r,s\in\mathbb{N},v_0,\ldots v_k\in W$ with $v_j=Y^j v_0$,$H v_j=(k-2j)v_j$,$Y v_k=0$,$X v_0=0$,
$\lambda_j=k-2j$,$X w_0=q_{(r-k)/2-1}$, $Y^{k+1}w_0=u_{(s+k)/2+1}$, $w_j=Y^j w_0$, $r,s\geq k$ for $j=\overline{0,k}$
and $H q_j=(r-2j)q_j$, $Y q_j=q_{j+1}$, $X q_j=j(r-j+1)q_{j-1}$ for $j=\overline{0,r}$ and
$H u_j=(s-2j)u_j$, $Y u_j=u_{j+1}$, $X u_j=j(s-j+1)u_{j-1}$ for $j=\overline{0,s}$ and
$Y q_r=0$, $X q_0=0$, $Y u_s=0$, $X u_0=0$,
$(q_l=0$ for $l<0$ or $l>r)$ and $(u_l=0$ for $l<0$ or $l>s)$.
(considering (2) , since H is not diagonalizable, for any $l\in\{0,\ldots,k\}$ we have $w\in W$, $\lambda\in\mathbb{C}$ such that $(H-\lambda\boldsymbol{I})w=v_l$ and the rest follows from (2) and the Jordan canonical structure).
By induction hypothesis and commutation relations we can reduce W to
$W=\mathrm{Sp}[X^l Y^n H^j(\{w_0,v_0\})]_{l,n,j\in\mathbb{N}}=\mathrm{Sp}[q_0,\ldots,q_r,u_0,\ldots,u_s,v_0,\ldots,v_k,w_0,\ldots,w_k]$
where $w_j=Y^j w_0$, $w_l=0$ for $(l<0$ or $l>k)$.
We must therefore have:

$$Y w_j=w_{j+1}+\gamma_j v_{j+1}+\alpha_j q_{(r-k)/2+j+1}+\beta_j u_{(s-k)/2+j+1}$$
$$X w_j=j(k-j+1)w_{j-1}+\bar{\gamma}_j v_{j-1}+\bar{\alpha}_j q_{(r-k)/2+j-1}+\bar{\beta}_j u_{(s-k)/2+j-1} \text{ for } j=\overline{0,k}$$

with $\gamma_k=\alpha_k=\bar{\gamma}_0=\bar{\beta}_0=0$, $\bar{\alpha}_0=\beta_k=1$
$\gamma_l=\bar{\gamma}_l=\alpha_l=\bar{\alpha}_l=\beta_l=\bar{\beta}_l=0$ for $(l<0$ or $l>k)$.
Hence

$$XY w_j=(j+1)(k-j)w_j+(\bar{\gamma}_{j+1}+(j+1)(k-j)\gamma_j)v_j+$$
$$+(\bar{\alpha}_{j+1}+((r-k)/2+j+1)((r+k)/2-j)\alpha_j)q_{(r-k)/2+j}+$$
$$+(\bar{\beta}_{j+1}+((s-k)/2+j+1)((s+k)/2-j)\beta_j)u_{(s-k)/2+j}$$
$$YX w_j=j(k-j+1)w_j+(j(k-j+1)\gamma_{j-1}+\bar{\gamma}_j)v_j+$$
$$+(j(k-j+1)\alpha_{j-1}+\bar{\alpha}_j)q_{(r-k)/2+j}+(j(k-j+1)\beta_{j-1}+\bar{\beta}_j)u_{(s-k)/2+j} \text{ for } j=\overline{0,k}$$

and it follows :

$$0=(\bar{\gamma}_{j+1}-\bar{\gamma}_j+(j+1)(k-j)\gamma_j-j(k-j+1)\gamma_{j-1}-1)v_j+$$
$$+(\bar{\alpha}_{j+1}-\bar{\alpha}_j+((r-k)/2+j+1)((r+k)/2-j)\alpha_j-j(k-j+1)\alpha_{j-1})q_{(r-k)/2+j}+ \quad (3)$$
$$+(\bar{\beta}_{j+1}-\bar{\beta}_j+((s-k)/2+j+1)((s+k)/2-j)\beta_j-j(k-j+1)\beta_{j-1}) \text{ for } j=\overline{0,k}$$

If $s = k$ and $r = k$, the $(q_j)_j$ and $(u_j)_j$ must be respective all independent of or parallel to the $(v_j)_j$. In all of this cases we have therefore a relation

$\bar{\gamma}_{j+1} - \bar{\gamma}_j + (j+1)(k-j)\gamma_j - j(k-j+1)\gamma_{j-1} - 1 = 0$ for $j = \overline{0,k}$ (4)
If $s > k$ or $r > k$ and if we can take
$j \in \{0, \ldots, k\}$ such that $\bar{\gamma}_{j+1} - \bar{\gamma}_j + (j+1)(k-j)\gamma_j - j(k-j+1)\gamma_{j-1} - 1 \neq 0$
then for this j we have
$\bar{\alpha}_{j+1} - \bar{\alpha}_j + ((r-k)/2 + j + 1)((r+k)/2 - j)\alpha - j(k-j+1)\alpha_{j-1} \neq 0$ (5) or
$\bar{\beta}_{j+1} - \bar{\beta}_j + ((s-k)/2 + j + 1)((s+k)/2 - j)\beta_j - j(k-j+1)\beta_{j-1} \neq 0$ (6).
If (5) is satisfied, applying X^{j+1} to (3) we obtain
$q_{(r-k)/2-1} \| u_{(s-k)/2-1}$ and so $\mathrm{Sp}[q_0, \ldots, q_r] = \mathrm{Sp}[u_0, \ldots, u_s]$, $s = r$, $q_l \| u_l$ for $l = \overline{0,r}$
and $v_j \| q_{(r-k)/2+j}$, $\mathrm{Sp}[v_0, \ldots, v_k] = \mathrm{Sp}[q_0, \ldots, q_r] = \mathrm{Sp}[u_0, \ldots, u_s]$
contradicting $(s > k$ or $r > k)$.
If (6) is satisfied, applying Y^{k-j+1} to (3) we obtain in a similar way a contradiction with ($s > k$ or $r > k$).
Therefore (4) is satisfied and taking the summation over j we obtain $k+1 = 0$ which contradicts $k \geq 0$. and so we complete the proof of $H_{\pm}$ diagonalizable .
Because H_+ and H_- commute and are diagonalizable, H_+ invariate any eigenspace of H_- , we have that we can find $V_i = \mathrm{Sp}[v_k^{(i)}]_{k=\overline{0,mi}}$ such that V_i are corresponding to irreducible representations of $g_- = \mathrm{Sp}[\bar{H}_-, \bar{X}_-, \bar{Y}_-]$ for $i = \overline{1,n}$,
$W = \bigoplus_{i=1}^{n} V_i$ and $v_k^{(i)}$ is an eigenvector of H_+ for any $k = \overline{0,m_i}$, $i = \overline{1,n}$.
For any $v \in \{v_k^{(i)}\}_{\substack{k=\overline{0,mi} \\ i=\overline{1,n}}} = S$ we can take $S_j(v) \subseteq S$, $j = 1,2$ such that
$X_+ v = \sum_{w \in S_1(v)} \alpha_w w$ with $\alpha_w \neq 0$ for any $w \in S_1(v)$ (7)
$Y_+ v = \sum_{w \in S_2(v)} \beta_w w$ with $\beta_w \neq 0$ for any $w \in S_2(v)$ (8)
Without difficulties, because H_+, X_+, Y_+ commute with H_-, X_-, Y_- , considering the minimal character of the chosen $S_i(v)$ (such that $\alpha_w \neq 0$ for any $w \in S_1(v)$ and $\beta_w \neq 0$ for any $w \in S_2(v)$), repeatedly applying X_+ , Y_+ , X_- , Y_- to change back and forward the levels of the H operators eigenvalues , we find for any $v \in S$ the values $r, p, k, l \in \mathbb{N}$ with $l \leq r$, $k \leq p$ such that for any $i = 1,2$, $w \in S_i(v)$ we have

$a_i(w)$, $c(w) \in W$ with

$w = Y_+^k a_i(w) = Y_-^l c(w)$, $Y_+^{p+1} a_i(w) = X_+ a_i(w) = Y_-^{r+1} c(w) = X_- c(w) = 0$,
$Y_+^p a_i(w) \neq 0$, $Y_-^r c(w) \neq 0$.

Also, if $S_1(v) \cup S_2(v) = \emptyset$ we have $d(v) \in W$, $h, t \in \mathbb{N}$, $h \leq t$ with
$Y_+ Y_-^j d(v) = X_+ Y_-^j d(v) = 0$ for $j = \overline{0,t}$, $v = Y_-^h d(v)$, $X_- d(v) = Y_-^{t+1} d(v) = 0$
$Y_-^t d(v) \neq 0$.

For $v \in S$ we can consider

$W(v)=\{u\in S\mid \text{exist } x_1,\ldots,x_f \text{ such that for any } i=\overline{1,f-1} \text{ exist } a,b\in\{1,2\} \text{ with } S_a(x_i)\cap S_b(x_{i+1})\neq\emptyset\ ,\ x_1=v\ ,\ x_f=u\}$

The above defined *r,p,l,k* depend on *v* and we denote $(r,p,l,k)=(r,p,l,k)(v)$ having $(r,p,l,k)(u)=(r,p,l,k)(v)$ for any $u\in W(v)$ and the space $R(v)=\mathrm{Sp}[X_+^\alpha Y_+^\beta H_+^\gamma X_-^\mu Y_-^\lambda H_-^\nu(S_j(u))]_{\substack{\alpha,\beta,\gamma,\mu,\lambda,\nu\in\mathbb{N}\\ u\in W(v)\ ,\ j=1,2}}$

is a direct sum of spaces of the form

$\mathrm{Sp}[Y_-^j Y_+^s a(u), X_-^i Y_+^s a(u)]_{j=\overline{0,r-l}\ ;\ i=\overline{0,l}\ ;\ s=\overline{0,p}}=K(u)$

with $u\in W(v)$, $(r,l,p)=(r,l,p)(v)$.

Also, if $R(v)\cap R(v')\neq\{0\}$ then $R(v)=R(v')$ and we have

$W=\mathrm{Sp}[\bigcup_{v\in S} R(v)]\oplus W_0$ where $W_0=\{u\in S\mid S_1(u)\cup S_2(u)=\emptyset\}$.

Thus *W* is a direct sum of irreducible representations of type *K*(*u*) and identical representations.

The *K*(*u*) representations can be indexed after (*r* /2 , *p/*2) : a spin *r*/2 representation for the (H_- , X_- , Y_-) generators and a spin *p*/2 representation for the (H_+ , X_+ , Y_+) generators.