Spin statistics theorem

Consider two completely identical systems a and b coupled through an interaction energy which is symmetric in the two systems.
If we first consider the two systems together, without including the interaction energy, the systems will have stationary energy eigenstates n, m as eigenfunctions $\Phi_n^a = \Phi_n^a(q^a)$, $\Phi_m^a = \Phi_m^b(q^b)$ and the total energy of the system is $\hat{H} = \hat{H}^a \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \hat{H}^b$ with eigenstates $(\Phi_n^a \otimes \Phi_m^b)_{n,m}$;
$\hat{H}^a \Phi_n^a = H_n^a \Phi_n^a$, $\hat{H}_m^b \Phi_m^b = H_m^b \Phi_m^b$; (q^a) , (q^b) spatial coordinates for a respective b system. The systems being identical we will have
$H_{nm} = H_n^a + H_m^b = H_m^a + H_n^b = H_{mn}$.
The udistorted by interaction total system is degenerated and each eigenstate is a doublet $(\Phi_n^a \otimes \Phi_m^b \ , \ \Phi_m^a \otimes \Phi_n^b)$ with the exception of that with $m = n$.
In every system distorted by interaction the degeneracy is broken: it corresponds to secular beats in which the energy of the two particle system pulses back and forth and the energy of the distorted system is given in first approximation by the time average of the interaction energy over the undistorted motion which will contain other terms that correspond to the transitions in which the systems a and b exchange places.
Since $(\Phi_n^a \otimes \Phi_m^b)_{n,m}$ is still a basis for the distorted total system Hilbert space, the distorted by interaction Hamiltonian will be:

$$\hat{H} = H^1(nm,nm) |\Phi_n^a\rangle \otimes |\Phi_m^b\rangle \langle \Phi_n^a| \otimes \langle \Phi_m^b| + H^1(mn,mn) |\Phi_m^a\rangle \otimes |\Phi_n^b\rangle \langle \Phi_m^a| \otimes \langle \Phi_n^b| +$$
$$H^1(nm,mn) |\Phi_n^a\rangle \otimes |\Phi_m^b\rangle \langle \Phi_m^a| \otimes \langle \Phi_m^a| \otimes \langle \Phi_n^b| + H^1(mn,nm) |\Phi_m^a\rangle \otimes |\Phi_n^b\rangle \langle \Phi_n^a| \otimes \langle \Phi_n^a| \otimes \langle \Phi_m^b| \ .$$

because the interaction energy is symmetric in the two systems, we have:

$$H^1(nm,nm) = H^1(mn,mn) \ , \ H^1(nm,mn) = H^1(mn,nm) \qquad (1)$$

Considering the (1) relations, diagonalizing $\hat{H}$ we obtain:

$$\hat{H} = (H^1(nm,nm) + H^1(nm,mn)) |\Phi_{+mn}\rangle \langle \Phi_{+mn}| +$$
$$+ (H^1(nm,nm) - H^1(nm,mn)) |\Phi_{-mn}\rangle \langle \Phi_{-mn}| \quad \text{where}$$
$$|\Phi_{+mn}\rangle = \frac{1}{\sqrt{2}} (|\Phi_n^a\rangle \otimes |\Phi_m^b\rangle + |\Phi_m^a\rangle \otimes |\Phi_n^b\rangle)$$
$$|\Phi_{-mn}\rangle = \frac{1}{\sqrt{2}} (|\Phi_n^a\rangle \otimes |\Phi_m^b\rangle - |\Phi_m^a\rangle \otimes |\Phi_n^b\rangle)$$

The eigenstates of $\hat{H}$ are now $(|\Phi_{+mn}\rangle)_{m,n}$, $(|\Phi_{-mn}\rangle)_{m,n}$ and a perturbation inducing transitions between different eigenstates of $\hat{H}$ can be written as $\hat{H}' = F \exp(-i\omega t) + F^+ \exp(i\omega t)$ where F and its adjoint operator F^+ are in general functions of $(\hat{p}^a, \hat{q}^a, \hat{p}^b, \hat{q}^b)$ which do not change under the interchange of the two systems and t is the time variable.
Then (according to Chap. Fermi's golden rule) the transition probability rate from a state $|i\rangle = |\Phi_{+mn}\rangle$ to a state $|f\rangle = |\Phi_{-m'n'}\rangle$ is proportional to

$$|\langle f|F|i\rangle|^2=\left|\int \frac{1}{2}(\Phi_m^a(q^a)\Phi_n^b(q^b)+\Phi_n^a(q^a)\Phi_m^b(q^b))\,F\,\left(\Phi_{n'}^{a+}(q^a)\Phi_{m'}^{b+}(q^b)-\right.\right.$$
$$\left.\left.-\Phi_{m'}^{a+}(q^a)\Phi_{n'}^{b+}(q^b)\right)d\,q^a\,d\,q^b\right|^2$$

The expression under the integral on the right side of the above relation changes sign when a and b are interchanged and so the integral vanishes and we conclude that transitions between $|\Phi_{+mn}\rangle$ states and $|\Phi_{-m'n'}\rangle$ similar between $|\Phi_{-mn}\rangle$ states and $|\Phi_{+m'n'}\rangle$ states cannot occur.

Thus the level spectrum of the combined, distorted by interaction system can be divided into two spectra which can never combine with one other:

the $(+)$: $|\Phi_m^a\rangle\otimes|\Phi_n^b\rangle+|\Phi_n^a\rangle\otimes|\Phi_m^b\rangle$ symmetric wave functions and

the $(-)$: $|\Phi_m^a\rangle\otimes|\Phi_n^b\rangle-|\Phi_n^a\rangle\otimes|\Phi_m^b\rangle$ antisymmetric wave functions .

The goal of the spin statistics theorem is to establish that only one of the two spectra (+) or (-) is allowed, namely (+) if the individual systems are bosonic (integer spin particles) and (-) if the individual systems are fermionic (half integer spin particles) and so bosons will obey to the Bose-Einstein statistics and fermions will obey to the Fermi-Dirac statistics (see Chap. Quantum statistical ensemble).

Consider now a quantum field particles system described by a particle field operator function $\hat{\Phi}=\hat{\Phi}(x)$, $x=(t,\vec{x})\in\mathbb{R}^4$ acting on a Hilbert space of state vectors containing an unique vacuum state $|0\rangle$ with $\Phi=(\Phi_\lambda)_\lambda$

$$\hat{\Phi}_\lambda(x)=\sum_s\int d^3\vec{k}(u_\lambda(k,s)\hat{b}(k,s)\exp(-ikx)+v_\lambda(k,s)\hat{d}^+(k,s)\exp(ikx)) \qquad (2)$$

where $k=(k_0,\vec{k})$, $k_0=\sqrt{\vec{k}^2+m^2}$ and $\hat{b}^+(k,s)$, $\hat{d}^+(k,s)$ / $\hat{b}(k,s)$, $\hat{d}(k,s)$ are creation/anihilation like operators acting on state vectors such that $\hat{b}|0\rangle=\hat{d}|0\rangle=0$ and $\hat{b}^+(k,s)|0\rangle$ is the spin index s and k - four-momentum state vector for the particle and $\hat{d}^+(k,s)|0\rangle$ being the same for the antiparticle.

We have an unitary representation of the inhomogeneous Lorentz group of Poincare transformations $x\rightarrow\Lambda x+a$ with $a\in\mathbb{R}^4$, $\Lambda\in SO^+(3,1)$, restricted Lorentz transformation such that:

$$U=U(a,\Lambda)\ ,\ \hat{\Phi}_\lambda(\Lambda x+a)=U(a,\Lambda)\hat{\Phi}_\lambda(x)U^+(a,\Lambda)$$

$U(a_1,\Lambda_1)U(a_2,\Lambda_2)=U(a_1+\Lambda_1 a_2,\Lambda_1\Lambda_2)$, $U(a,\boldsymbol{I})=\exp(i\hat{p}a)$, $U|0\rangle=|0\rangle$ where $\hat{p}$ is the four-momentum operator acting on state vectors

(we have indeed $\hat{\Phi}_\lambda^+(x+a)=\int d^3\vec{k}\,u_\lambda^*(k,s)\exp(ika)\hat{b}^+(k,s)\exp(ikx)|0\rangle=$
$=\int d^3\vec{k}\exp(i\hat{p}a)\hat{b}^+(k,s)\exp(ikx)|0\rangle=U(a,\boldsymbol{I})\hat{\Phi}_\lambda^+(x)U^+(a,\boldsymbol{I})|0\rangle$).

Φ is transforming under some finite dimensional irreducible representation of the restricted Lorentz group : $x\rightarrow\Lambda x=x'$ as $\hat{\Phi}_\lambda(x)\rightarrow\hat{\Phi}'_\lambda(x')=S_{\lambda\mu}(\Lambda)\hat{\Phi}_\mu(x)$.

We also consider an invariant space of test functions, $f=(f^\mu(x))_\mu$ such that we we have the transformation

$f^{\mu}(x) \to f^{\mu}_{\Lambda}(x') = f^{\lambda}(x) S_{\lambda\mu}(\Lambda^{-1})$ so that the test field operator defined as

$\hat{\Phi}(f) = \int d^4 x f^{\mu}(x) \hat{\Phi}_{\mu}(x)$ is invariant under $x \to \Lambda x = x'$.

Suppose we want to interchange two space-like separated particles generated by the field at the points x , y in space-time and test the system with test functions $f^{\lambda}(x)$ and $g^{\mu}(y)$. We test therefore the system at the neighbourhoods of two space-like separated points 1 and 2 generated by the field in points *x* respective *y* using arbitrary test functions *f* (*x*) , *g* (*y*). For interchanging the particles we must have a trajectory in space-time, expressing the interchanging process, from *x* to *y*. Since according to relativity no process from a point A to apoint B exist if B is not in the future light-cone of A (or A is in the past light-cone of B) we must therefore perform a measuring process of the vacuum at point *y* applying first to the vacuum state the corresponding test field operator and then we must find a way to conduct the process from *y* to the space-like separated *x*. Because *x* and *y* are space-like separated and we want the time to run forward when we arrive at *x* we must conduct the process from *y* to a point *z* which is in the past light-cone of *y* and in the past light-cone of *x*. Then we conduct the process from *z* to *x*, *x* being in the future light-cone of *z*. Because from *y* to *z* we move backwards in time, for having a field effect of positive energy we must consider the term in exp(*iky*) in the field expression (2) at *y* and since $\hat{b}|0\rangle = \hat{d}|0\rangle = 0$ we must take at y the effect of $g^{\mu *}(y) \hat{\Phi}^{+}_{\mu}(y)$ as correspondng test field operator on the vacuum state $|0\rangle$. Φ^{+} is the antiparticle field, which according to interpretation is the particle moving backwards in time, that is the field creates, moving from the future proximity of *y* (where the particle 2 is supposed to be) a particle with positive energy. Now we can conduct further the measuring process by applying the field operator $f^{\lambda}(x) \hat{\Phi}_{\lambda}(x)$ on $g^{\mu *}(y) \hat{\Phi}^{+}_{\mu}(y)|0\rangle$, that is the field anihilates moving from the past proximity of *x* (where the particle 1 is supposed to disappear) a particle with positive energy.

The required expectation value of the measuring process is therefore $\langle 0| f^{\lambda}(x) \hat{\Phi}_{\lambda}(x) \hat{\Phi}^{+}_{\mu}(y) g^{\mu *}(y)|0\rangle$.

If we switch now particles and test first at *x* and then at *y* we move from *x* to a point *z'* in the future light-cone of *x* and in the future light cone of *y* and from *z'* , which is in the past light-cone of *y* , getting the expectation value $\langle 0| g^{\mu *}(y) \hat{\Phi}^{+}_{\mu}(y) \hat{\Phi}_{\lambda}(x) f^{\lambda}(x)|0\rangle$.

For $\xi = y - x$ we have $\langle 0|\hat{\Phi}^{+}_{\mu}(y) \hat{\Phi}_{\lambda}(x)|0\rangle =$

$$= \langle 0| U(y, \boldsymbol{I}) \hat{\Phi}^{+}_{\mu}(0) U^{+}(y, \boldsymbol{I}) U(y, \boldsymbol{I}) \hat{\Phi}_{\lambda}(x-y) U^{+}(y, \boldsymbol{I})|0\rangle =$$

$$= \langle 0|\hat{\Phi}^{+}_{\mu}(0) \hat{\Phi}_{\lambda}(-\xi)|0\rangle = H_{\mu\lambda}(\xi) = \langle 0|\hat{\Phi}^{+}_{\mu}(0) \exp(-i \hat{p} \xi) \hat{\Phi}_{\lambda}(0)|0\rangle.$$

Let $K_{\mu\lambda}(p) = \int d^4 \xi \exp(i p \xi) \langle 0|\hat{\Phi}^{+}_{\mu}(0) \exp(-i \hat{p} \xi) \hat{\Phi}_{\lambda}(0)|0\rangle$.

We can always take a complete orthonormate system of four-momentum state vectors $(|\psi\rangle)_{\psi}$ with $\hat{p}|\psi\rangle = p_{\psi}|\psi\rangle$, $\sum_{\psi} |\psi\rangle\langle\psi| = \boldsymbol{I}$ and $p_{\psi 0} \geq 0$, $p^2_{\psi} \geq 0$ for any permissible state vector $|\psi\rangle$ and we will have:

$$K_{\mu\lambda}(p)=\sum_{\psi}\int d^4\xi \exp(i p\xi)\langle 0|\hat{\Phi}_{\mu}^{+}(0)\exp(-i\hat{p}\xi)|\psi\rangle\langle\psi|\hat{\Phi}_{\lambda}(0)|0\rangle=$$
$$=\sum_{\psi}(2\pi)^4\delta^4(p-p_{\psi})\langle\psi|\hat{\Phi}_{\lambda}(0)|0\rangle\langle 0|\hat{\Phi}_{\mu}^{+}(0)|\psi\rangle$$
$$H_{\mu\lambda}(\xi)=\int\frac{1}{(2\pi)^4}\exp(-i p\xi)K_{\mu\lambda}(p)d^4p=$$
$$=\sum_{\psi}\exp(-i p_{\psi}\xi)\langle\psi|\hat{\Phi}_{\lambda}(0)|0\rangle\langle 0|\hat{\Phi}_{\mu}^{+}(0)|\psi\rangle \qquad (3).$$

For $\hat{\Phi}_{\lambda}(0)|0\rangle=\sum_{\psi}c_{\psi}|\psi\rangle$, $\langle 0|\hat{\Phi}_{\mu}^{+}(0)=\sum_{\psi}d_{\psi}\langle\psi|$ since $(|\psi\rangle)_{\psi}$ is a complete orthonormal system we have $\sum_{\psi}|c_{\psi}||d_{\psi}|\le(\langle 0|\hat{\Phi}_{\mu}^{+}(0)\hat{\Phi}_{\mu}(0)|0\rangle\langle 0|\hat{\Phi}_{\lambda}^{+}\hat{\Phi}_{\lambda}(0)|0\rangle)<\infty$ and so the series on the right side of (3) is absolute convergent and uniform absolute convergent with respect to $z\in\{\xi-i\eta\in\mathbb{C}^4|\eta_0\ge 0\}$ (z on the place of ξ variable in (3)) .

Hence $H_{\mu\lambda}(\xi)=\lim_{\eta\to 0\ ,\ \eta 0\ge 0}H_{\mu\lambda}(\xi-i\eta)$ and for $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}^4$ $\alpha!=\alpha_0!\,\alpha_1!\,\alpha_2!\,\alpha_3!$ we have if $\eta_0>0$ that $p_{\psi}\eta\ge 0$ and so:

$$\sum_{\alpha}\sum_{\psi}\frac{|(p_{\psi})^{\alpha}|}{\alpha!}|\exp(-i p_{\psi}(\xi-i\eta))||\zeta^{\alpha}||\langle\psi|\hat{\Phi}_{\lambda}(0)|0\rangle\langle 0|\hat{\Phi}_{\mu}^{+}(0)|\psi\rangle|\le$$
$$\le\exp(4p_{\psi 0}\|\zeta\|)\sum_{\psi}|c_{\psi}||d_{\psi}| \text{ for any } \zeta\in\mathbb{C}^4.$$

Therfore we have an analytic function on $D=\{\xi-i\eta\in\mathbb{C}^4|\eta_0>0\}\cap\{z\in\mathbb{C}^4|z^2\ne 0\}$with $z=(z_0,z_1,z_2,z_3)$, $z^2=z_0^2-z_1^2-z_2^2-z_3^2$, $H_{\mu\lambda}(z)=\sum_{\psi}\exp(-i p_{\psi}z)c_{\psi}d_{\psi}$ and $H_{\mu\lambda}(\xi)$ is the boundary value of an analytic function on *D*.

We have for any $\Lambda\in SO^{+}(3,1)$ that $H_{\mu\lambda}(\Lambda\xi)=\langle 0|\hat{\Phi}_{\mu}^{+}(\Lambda 0)\hat{\Phi}_{\lambda}(-\Lambda\xi)|0\rangle=$ $=\langle 0|U\hat{\Phi}_{\mu}^{+}(0)U^{+}U\hat{\Phi}_{\lambda}(-\xi)U^{+}|0\rangle=H_{\mu\lambda}(\xi)$ where $U=U(0,\Lambda)$.

Hence for any $\Lambda\in SO^{+}(3{,}1)$, $z\in D$ we have $H_{\mu\lambda}(\Lambda z)=H_{\mu\lambda}(z)$(because if $z\in D$ then $\Lambda z\in D$.

Let $(J_k,K_k)_{k=\overline{1,3}}$ the generators of $SO^{+}(3{,}1)$ (see Chap. Representations of the restricted Lorentz group). It is easy to see that for any $\chi\in\mathbb{C}$, $z\in D$, if $\Lambda=\exp(\chi K_3)$ satisfies $\Lambda z\in D$ then exists a continuous path $\gamma:[0{,}1]\to\mathbb{C}$ with $\gamma(0)=0$, $\gamma(1)=\chi$ such that for $\Lambda_t=\exp(\gamma(t)K_3)$ we have $\Lambda_t z\in D$ for any $t\in[0{,}1]$ and therefore, by analytic continuation we obtain

$$H_{\mu\lambda}(\Lambda z)=H_{\mu\lambda}(z) \text{ for } \Lambda=\exp(\chi K_3)\ ,\ z\in D\ ,\ \Lambda z\in D\ .$$

Consider $Q=\{\exp(\theta_1J_1+\theta_2J_2+\theta_3J_3)|\theta_i\in\mathbb{C}\ ,\ i=\overline{1{,}3}\}=S$.

Obviously $Qz\in D$ for any $z\in D$ and by analytic continuation $H_{\mu\lambda}(Qz)=H_{\mu\lambda}(z)$.

Thus for given $z=(z_0, z_1, z_2, z_3)\in D$ we find $Q\in S$ with $Qz=z'=(z'_0, 0, 0, z'_3)$ $z'^2=z^2$, $z'\in D$ and we can choose $\chi\in\mathbb{C}$ such that for $\Lambda=\exp(\chi K_3)$ we have $\Lambda z'=(\sqrt{z^2}, 0, 0, 0)$ where the square root $\sqrt{z^2}$ is taken with positive imaginary part. We will have $H_{\mu\lambda}(\Lambda z')=H_{\mu\lambda}(z')=H_{\mu\lambda}(Qz)=H_{\mu\lambda}(z)$ and so $H_{\mu\lambda}(z)=H_{\mu\lambda}(\sqrt{z^2}, 0, 0, 0)$ and since we can verify that $\{z^2|z\in D\}\supset\mathbb{C}\setminus\mathbb{R}_+$ and so

$H_{\mu\lambda}$ can be analytically continuated on $B=\{\xi\in\mathbb{C}^4|\,\xi^2\in\mathbb{C}\setminus\mathbb{R}_+\}$.

Since $\langle 0|\hat{\Phi}^+_\mu(y)\hat{\Phi}_\lambda(x)|0\rangle$ must be Lorentz invariant it follows that for any $\Lambda\in SO^+(3,1)$, $U=U(0,\Lambda)$ we have $H_{\mu\lambda}(\Lambda\xi)=S^*_{\mu\mu'}(\Lambda)S_{\lambda\lambda'}(\Lambda)H_{\mu'\lambda'}(\xi)$ (4) where S^* is the complex conjugate of S.

According to Chap. Representations of the restricted Lorentz group (final) we have that S is an irreducible (j_1, j_2) representation so that

$$j_1, j_2\in\frac{1}{2}\mathbb{N}\ ,\ \Lambda=\exp((-i\vec{\theta}-\vec{\chi})\frac{1}{2}(i\vec{J}-\vec{K}))\exp((-i\theta+\vec{\chi})\frac{1}{2}(i\vec{J}+\vec{K}))$$

$$S(\Lambda)=\exp((-i\vec{\theta}-\vec{\chi})\vec{M}_+)\exp((-i\vec{\theta}+\vec{\chi})\vec{M}_-)\ ,\ M_{+3}=\frac{1}{2}H_+\ ,\ M_{-3}=\frac{1}{2}H_-$$

H_+ having the spectrum $(-2j_1, -2j_1+2, \dots, 2j_1)$ and
H_- having the spectrum $(-2j_2, -2j_2+2, \dots, 2j_2)$.

Because for any $\Lambda=\exp((-i\theta-\chi)\frac{1}{2}(iJ_3-K_3))\exp((-i\theta+\chi)\frac{1}{2}(iJ_3+K_3));\theta,\chi\in\mathbb{C}$ we have $(\Lambda z)^2=z^2$,by analytic continuation on θ,χ variables, we can take in (4)

$$\Lambda=\exp(-i\pi(iJ_3-K_3))\ ,\ S(\Lambda)=\exp(-i\pi H_+)\ ,\ S^*(\Lambda)=\exp(i\pi H^*_-).$$

iJ_3-K_3 has eigenvalues ± 1 and is diagonalizable and so $\Lambda=-\boldsymbol{I}$. H_+ and H^*_- are also diagonalizable (see Chap. Repres. of the restricted Lorentz group (final)) and so $\exp(-i\pi H_+)=\exp(2j_1 i\pi)\boldsymbol{I}$, $\exp(i\pi H^*_-)=\exp(2j_2 i\pi)\boldsymbol{I}$.

Hence $H_{\mu\lambda}(-\xi)=\exp(2(j_1+j_2)i\pi)H_{\mu\lambda}(\xi)$ for any space-like ξ because for $\xi^2<0$ we have $\xi\in B$. Thus for bosons $j_1+j_2\in\mathbb{N}$ and we have $H_{\mu\lambda}(-\xi)=H_{\mu\lambda}(\xi)$ and for fermions $j_1+j_2\in\frac{1}{2}+\mathbb{N}$ and we have $H_{\mu\lambda}(-\xi)=-H_{\mu\lambda}(\xi)$.

Testing the interchanging of particles in space-like separated points x , y with $f^{\lambda'}(x')\to\delta^4(x'-x)\delta_{\lambda\lambda'}$, $g^{\mu'}(y')\to\delta^4(y'-y)\delta_{\mu\mu'}$ with x' , y' variable and $\to$ convergence in distributions space, since we have already proven the splitting of the level spectrum in (+) and (-) spectra with no transitions between them, we will have for the expectation values, considering the commuting/anticommuting for the individual wave functions of the respective spectra (+)/(-), one of the relations:
$\langle 0|\hat{\Phi}^+_\mu(y)\hat{\Phi}_\lambda(x)|0\rangle=\pm\langle 0|\hat{\Phi}_\lambda(x)\hat{\Phi}^+_\mu(y)|0\rangle$.

Proving the spin statistics theorem is therefore reduced to verify that “wrong” commutation relations cannot take place.

Taking $F_{\mu\lambda}(-\xi)=\langle 0|\hat{\Phi}_\mu(x)\hat{\Phi}^+_\lambda(y)|0\rangle=\langle 0|\hat{\Phi}_\mu(0)\hat{\Phi}^+_\lambda(\xi)|0\rangle$ the 'wrong' commutation relation is for both bosonic and fermionic case

$F_{\lambda\mu}(\xi)+H_{\mu\lambda}(\xi)=0$ for $\xi^2<0$ (5)

(because $H_{\mu\lambda}(-\xi)=(-1)^{2(j1+j2)}H_{\mu\lambda}(\xi)$ for a (j_1,j_2) representation if $\xi^2<0$.

Let for $f=f(x)$, $\bar{f}=f(-x)$ and we will have:

$\|\hat{\Phi}(f)|0\rangle\|^2=\int f^{\mu*}(y)\langle 0|\hat{\Phi}_\mu^+(y)\hat{\Phi}_\lambda(x)|0\rangle f^\lambda(x)d^4x\,d^4y=$

$=\int f^{\mu*}(y)H_{\mu\lambda}(y-x)f^\lambda(x)d^4x\,d^4y$

$\|\hat{\Phi}^+(\bar{f})|0\rangle\|=\int f^\mu(-y)\langle 0|\hat{\Phi}_\mu(y)\hat{\Phi}_\lambda^+(x)|0\rangle f^{\lambda*}(-x)d^4x\,d^4y=$

$=\int f^\mu(y)\langle 0|\hat{\Phi}_\mu(x)\hat{\Phi}_\lambda^+(y)|0\rangle f^{\lambda*}(x)=\int f^{\mu*}(y)F_{\lambda\mu}(y-x)f^\lambda(x)d^4x\,d^4y$

where for the last equality we have taken the complex conjugate of the integrand considering the fact that the left side is real , being a squared norm.

Taking for $y-x=\xi$, $\xi^2<0$: $f^\lambda(x')=\alpha_\lambda\delta^4(x'-x)+\beta_\lambda\delta^4(x'-y)$ with arbitrary α_λ , β_λ , from (5) follows now $\|\hat{\Phi}(f)|0\rangle\|=\|\hat{\Phi}^+(\bar{f})|0\rangle\|=0$ and so, α,β being arbitrary we obtain $\hat{\Phi}_\lambda(x)|0\rangle=\hat{\Phi}_\lambda^+(x)|0\rangle=0$ for any $\lambda,x,$ which leads, considering (2) to the conclusion that $\hat{\Phi}=0$ and thus the 'wrong' commutation relations cannot take place or the field vanishes, which proves the spin statistics theorem.

(indeed, multiplying for example $\hat{\Phi}_\lambda^+(x)|0\rangle=0$ with $\exp(-ik\,x)$ for given k and integrating over $x\in\mathbb{R}^4$ it follows $\sum_s\delta(k_0-\sqrt{\vec{k}^2+m^2})u_\lambda^*(k,s)\hat{b}^+(k,s)|0\rangle=0$ (6)

so that applying to (6) $\hat{b}(k,s')$, since $\hat{b},\hat{b}^+$ are anihilation/creation like operators and satisfy an commutation/anticommutation rule

$\hat{b}(k,s)\hat{b}^+(k',s')\pm\hat{b}^+(k',s')\hat{b}(k,s)=\delta^3(\vec{k}'-\vec{k})\delta_{ss'}$

we obtain $u_\lambda^*(k,s)=0$ for any λ,k,s with $k_0=\sqrt{\vec{k}^2+m^2}$)

References

[1] A. Zee, Quantum Field Theory in a Nutshell, Second Edition, Princeton University Press – Princeton and Oxford
[2] Ian Duck, ECG Sudarshan, Pauli and the Spin-statistics Theorem, World Scientific
[3] Alexander Kirillov, Jr. Introduction to Lie Groups and Lie Algebras, URL: http://www.math.sunysb.edu/~kirillov