Mott scatering. Scattering of a spin ½ particle on a spinless charged particle. Spin Hall effect

As before we take the speed of light in vacuum constant $c = 1$, reduced Planck constant $\hbar = 1$, electric permittivity of vacuum $\varepsilon = 1$ (by suitable choosing of measuring units for time , length and electric charge).
Let p be the incoming four-momentum of the charged spin ½ particle with charge q_1 and mass m and k the incoming four-momentum of the spinless particle having charge q_2 and mass μ . The mass μ is supposed to be much larger than the mass m and so we can consider the scattering in the mass center frame of the particles which in this case can be assimilated to the lab frame where the heavier particle is at rest.
Then according to Feynman rules (see Chap. Feynman amplitudes and lattice gauge theory) , the Feynman amplitude of the scattering process at q_1 q_2 first order is

$$A=(2\pi)^4 M\,\delta^4(p+k-p'-k')\ ,$$

$$M=-i q_1 q_2 \bar{u}(p')\gamma^{\mu}u(p)\frac{1}{(p-p')^2}(k+k')_{\mu}\ ,$$

where p', k' are the outgoing four-momenta of the fermion respective the spinless particle.
In the center of mass frame we have $\vec{p}+\vec{k}=\vec{p}\,'+\vec{k}\,'=0$ and energy conservation leads to $E=p_0+k_0=p'_0+k'_0$, $\|\vec{p}\|=\|\vec{k}\|=\|\vec{p}\,'\|=\|\vec{k}\,'\|=r$,

$$r=\frac{1}{2E}((E^2-(m+\mu)^2)(E^2-(\mu-m)^2))^{1/2} \text{ with } p_0=p'_0=\sqrt{r^2+m^2}\ ,\ k_0=k'_0=\sqrt{r^2+\mu^2}\ .$$

According to Chap. Canonical quantization of a scalar field, decay rate and cross section, taking as in Chap. Feynman amplitudes and lattice gauge theory, for the electron field the normalization E_p/m instead of $2\omega(p)$ in the cross section formula, we will have a differential cross section given by

$$\frac{d\sigma}{d\Omega}=\frac{1}{|v_1-v_2|}\frac{4m^2}{4k_0p_0}\frac{1}{(2\pi)^2}\frac{2(k_0+p_0)r}{8(k_0+p_0)^2}|M|^2$$

where in the mass center frame $v_1=\frac{r}{p_0}$, $v_2=-\frac{r}{k_0}$ and so

$$\frac{d\sigma}{d\Omega}=\frac{m^2}{E^2}\frac{1}{(4\pi)^2}|M|^2\ .$$

As we proved in Chap. Anomalous magnetic moment of the electron we have the Gordon decomposition

$$\bar{u}(p')\gamma^{\mu}u(p)=\frac{1}{2m}\bar{u}(p')((p'+p)^{\mu}+i\sigma^{\mu\nu}(p'-p)_{\nu})u(p) \text{ where } \sigma^{\mu\nu}=\frac{1}{2}i[\gamma^{\mu},\gamma^{\nu}]\ .$$

Thus

$$M=\frac{-iq_1q_2}{2m}\bar{u}(p')((p'+p)(k'+k)+i(p'-p)_{\nu}(k'+k)_{\mu}\sigma^{\mu\nu})u(p)\frac{1}{(p'-p)^2}\ .$$

In the mass center frame we have
$(p'+p)(k'+k)=(p'+p)(2k+p-p')=2k(p+p')=4k_0p_0+2\vec{p}(\vec{p}+\vec{p}')=$
$=4k_0p_0+4r^2\cos^2(\theta/2)$ considering that $p^2=p'^2=m^2$ and θ is the scattering deflection angle between $\vec{p}$ and $\vec{p}'$: $\vec{p}\cdot\vec{p}'=r^2\cos(\theta)$, $\vec{p}\times\vec{p}'=e_3r^2\sin(\theta)$, where $e_j=(\delta_{ij})_{i=\overline{1,3}}$ for $j=\overline{1,3}$.

$i(p'-p)_\nu(k+k')_\mu\sigma^{\mu\nu}=-((p'-p)_\nu(k+k')_0-(p'-p)_0(k+k')_\nu)\gamma^0\gamma^\nu-$
$-i((\vec{p}'-\vec{p})\times(\vec{k}+\vec{k}'))\cdot\vec{\Sigma}=$
$=2k_0(\gamma^0\not{p}-\not{p}'\gamma^0)-4k_0p_0-2i(\vec{p}\times\vec{p}')\cdot\vec{\Sigma}$, where as usual $\not{p}=\gamma^\nu p_\nu$,

$\vec{\Sigma}=\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$, $\vec{\sigma}=(\sigma_i)_{i=\overline{1,3}}$ are the Pauli matrices.

$(p'-p)^2=2m^2-2(m^2+r^2)+2r^2\cos(\theta)=-4r^2\sin^2(\theta/2)$.

Because $\bar{u}(p')\not{p}'=m\bar{u}(p')$, $\not{p}u(p)=mu(p)$ we obtain

$$M=\frac{iq_1q_2}{m}\Big(2mk_0\bar{u}(p')\gamma^0u(p)+$$
$$+2r^2\cos^2(\theta/2)\bar{u}(p')u(p)-ir^2\sin(\theta)\bar{u}(p')\Sigma_3u(p)\Big)\frac{1}{4r^2\sin^2(\theta/2)} \; .$$

Consider now a flat conductor plate in which exists a flux of electrons $\vec{j}$ and we always measure the spin in direction $\vec{n}$. We have two relevant cases:
1) $\vec{n}=e_3$, $\vec{j}\| e_1$ and (e_1,e_2) is the plane of the conductor plate;
2) $\vec{n}=e_3$, the conductor plate plane is (e_2,e_3) and $\vec{j}\|(0,\cos(\varphi),\sin(\varphi))$.

In the 1) case we measure the spin normal to the motion of electrons and in 2) case we measure the spin along a direction contained in the plane in which the electrons are constrained to move.

Let $\varepsilon,\varepsilon'\in\{\pm\}$ the spin polarizations of the incoming respective outgoing electron in the scattering process. The charge carriers (electrons with mass *m* and charege *e* = - |*e*|) have spin up or down states along e_3 and can scatter on impurities from the cristal lattice grid, impurities that build a network of heavy charged diffusion centers in the way of motion for the electrons. A diffusion center is considered to have charge *q* and mass μ .

For a scattering on a diffusion center, the $\vec{p}'$ outgoing moment of the electron is constrained to be in the (e_1 , e_2) plane in the 1) case and in the (e_2 , e_3) plane in the 2) case and we will have $\vec{p}\times\vec{p}'\| e_3$, $\vec{p}\| e_1$ in the 1) case and $\vec{p}\times\vec{p}'\| e_1$,
$\vec{p}=r(0,\cos(\varphi),\sin(\varphi))$ in the 2) case.

We take first the 1) case:

Let $B=B(\chi,e_1)=\begin{pmatrix}\cosh\chi & -\sinh\chi & 0 & 0\\ -\sinh\chi & \cosh\chi & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}=\exp(-\chi K_1)$

where $p=(p_0,-r,0,0)=(p^\alpha)_\alpha$ as a column vector with

$$B\begin{pmatrix}m\\0\\0\\0\end{pmatrix}=p \quad , \quad \sinh\chi=\frac{r}{m} \quad , \quad \cosh\chi=\frac{p_0}{m}$$

$$R=R(\theta,e_3)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix}=\exp(\theta J_3) \ ,$$

K_l, J_l , $l=\overline{1,3}$ Lorentz group generators.

(see Chap. Representations of the rotations group and of the restricted Lorentz group)

Then for $P=\cosh(\frac{\chi}{2})\boldsymbol{I}+\sinh(\frac{\chi}{2})\gamma^1\gamma^0$,

$Q=\cos(\frac{\theta}{2})\boldsymbol{I}-i\sin(\frac{\theta}{2})\Sigma_3=\cos(\frac{\theta}{2})\boldsymbol{I}+\sin(\frac{\theta}{2})\gamma^1\gamma^2$,

$u_\varepsilon=\begin{pmatrix}w_\varepsilon\\0\\0\end{pmatrix}$, $w_+=\begin{pmatrix}1\\0\end{pmatrix}$, $w_-=\begin{pmatrix}0\\1\end{pmatrix}$ we will have:

$u(p)=P u_\varepsilon$, $u(p')=Q P u_{\varepsilon'}$.

To compute M for the ε , ε' spin polarizations of the incoming respective outgoing electrons we have to compute

$\bar{u}(p')\gamma^0 u(p)=u_{\varepsilon'}^+ P^+ Q^+ P u_\varepsilon=H$,

$\bar{u}(p')u(p)=u_{\varepsilon'}^+ P^+ Q^+ \gamma^0 P u_\varepsilon=G$,

$\bar{u}(p')\Sigma_3 u(p)=u_{\varepsilon'}^+ P^+ Q^+ \gamma^0\Sigma_3 P u_\varepsilon=K$.

After some calculus we obtain :

$$H=\delta_{\varepsilon\varepsilon'}(\cos(\frac{\theta}{2})\cosh\chi+i\varepsilon\sin(\frac{\theta}{2})) \ ,$$

$$G=\delta_{\varepsilon\varepsilon'}(\cos(\frac{\theta}{2})+i\varepsilon\sin(\frac{\theta}{2})\cosh\chi) \ ,$$

$$K=\delta_{\varepsilon\varepsilon'}(\varepsilon\cos(\frac{\theta}{2})\cosh\chi+i\sin(\frac{\theta}{2})) \ ,$$

$$M=\frac{ieq}{2mr^2\sin^2(\theta/2)}\delta_{\varepsilon\varepsilon'}((k_0 p_0+r^2)\cos(\frac{\theta}{2})+i\varepsilon m k_0\sin(\frac{\theta}{2})) \ ,$$

$$\frac{d\sigma}{d\Omega}=\left(\frac{\alpha}{2E}\right)^2\frac{1}{r^4\sin^4(\theta/2)}((k_0 p_0+r^2)^2\cos^2(\frac{\theta}{2})+m^2k_0^2\sin^2(\frac{\theta}{2})) \ .$$

$$(k_0 p_0+r^2)^2\cos^2(\frac{\theta}{2})+m^2 k_0^2\sin^2(\frac{\theta}{2})=k_0^2 p_0^2-k_0^2 r^2\sin^2(\frac{\theta}{2})+2r^2 k_0 p_0\cos^2(\frac{\theta}{2})+$$
$$+r^4\cos^2(\frac{\theta}{2}) \ .$$

For $\frac{r}{m},\frac{m}{\mu}\in O(\epsilon)$ in a non-relativistic approach we have $\frac{r^2 k_0 p_0}{(k_0 p_0)^2}=\frac{r}{m}\frac{r}{p_0}\frac{m}{k_0}\in O(\epsilon^3)$ and so disposing of the $O(\varepsilon^3)$ terms we obtain

$$\frac{d\sigma}{d\Omega}=\left(\frac{\alpha k_0 p_0}{2E}\right)^2\frac{1}{r^4\sin^4(\theta/2)}(1-v^2\sin^2(\frac{\theta}{2}))=\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}}(1-v^2\sin^2(\frac{\theta}{2}))$$ with $v=\frac{r}{p_0}$ the incoming velocity and $\alpha=\frac{eq}{4\pi}$.

As we can see, in the 1) case spin flipping is not allowed during the scattering process and the amplitude depends on the incoming spin polarization but the cross section does not depend on spin polarizations (since $\varepsilon^2=1$).

However if there is a spin-orbit coupling between the electron and the diffusion center we expect the differential cross section to be spin polarization dependent since the spin-orbit coupling involves the magnetic moment of the electron and so different spin polarization electrons will be scattered at different angles. This fact can arise by higher order scattering Feynman diagrams since the amplitudes have to be added with the first order diagrams amplitudes and so ε comes in in the squared absolute amplitude.

In the case 2) we take

$$B=\exp(-\chi K_2) \ , \ R=\exp(\theta J_1) \ , \ S=\exp(\varphi J_1) \ ,$$

$$P=\cosh(\frac{\chi}{2})\boldsymbol{I}+\sinh(\frac{\chi}{2})\gamma^2\gamma^0 \ , \ Q=\cos(\frac{\theta}{2})\boldsymbol{I}-i\sin(\frac{\theta}{2})\Sigma_1=\cos(\frac{\theta}{2})\boldsymbol{I}+\sin(\frac{\theta}{2})\gamma^2\gamma^3 \ ,$$

$$C=\cos(\frac{\varphi}{2})\boldsymbol{I}-i\sin(\frac{\varphi}{2})\Sigma_1=\cos(\frac{\varphi}{2})\boldsymbol{I}+\sin(\frac{\varphi}{2})\gamma^2\gamma^3$$

having $u(p)=CPu_\varepsilon$, $u(p')=QCPu_{\varepsilon'}$ and we have to compute

$$\bar{u}(p')\gamma^0 u(p)=u_{\varepsilon'}^+P^+C^+Q^+CPu_\varepsilon=u_{\varepsilon'}^+P^+Q^+Pu_\varepsilon=\tilde{H}$$
$$\bar{u}(p')u(p)=u_{\varepsilon'}^+P^+C^+Q^+\gamma^0 CPu_\varepsilon=u_{\varepsilon'}^+P^+Q^+\gamma^0 Pu_\varepsilon=\tilde{G}$$
$$\bar{u}(p')\Sigma_1 u(p)=u_{\varepsilon'}^+P^+C^+Q^+\gamma^0\Sigma_1 CPu_\varepsilon=u_{\varepsilon'}^+P^+Q^+\gamma^0\Sigma_1 Pu_\varepsilon=\tilde{K} \ .$$

We obtain

$$\tilde{H}=\delta_{\varepsilon\varepsilon'}\cos(\frac{\theta}{2})\cosh\chi+i\,\delta_{-\varepsilon'\varepsilon}\sin(\frac{\theta}{2})$$
$$\tilde{G}=\delta_{\varepsilon\varepsilon'}\cos(\frac{\theta}{2})+i\,\delta_{-\varepsilon'\varepsilon}\sin(\frac{\theta}{2})\cosh\chi$$
$$\tilde{K}=\delta_{-\varepsilon\varepsilon'}\cos(\frac{\theta}{2})\cosh\chi+i\,\delta_{\varepsilon'\varepsilon}\sin(\frac{\theta}{2})$$

$$M=\frac{ieq}{2mr^2\sin^2(\frac{\theta}{2})}\left(\delta_{\varepsilon\varepsilon'}(p_0k_0\cos(\frac{\theta}{2})+r^2\cos(\frac{\theta}{2}))+\delta_{-\varepsilon'\varepsilon}imk_0\sin(\frac{\theta}{2})\right)$$

$$\frac{d\sigma}{d\Omega}=\left(\frac{\alpha}{2E}\right)^2\frac{1}{r^4\sin^4(\frac{\theta}{2})}\left(\delta_{\varepsilon\varepsilon'}(k_0p_0+r^2)^2\cos^2(\frac{\theta}{2})+\delta_{-\varepsilon'\varepsilon}m^2k_0^2\sin^2(\frac{\theta}{2})\right)$$

Considering again the non-relativistic case $\frac{r}{m},\frac{m}{\mu}\in O(\epsilon)$ and disposing of all the $O(\epsilon^3)$ smaller terms we will have:

$$\frac{d\sigma}{d\Omega}=\left(\frac{\alpha k_0p_0}{2E}\right)^2\frac{1}{r^4\sin^4(\frac{\theta}{2})}\left(\delta_{\varepsilon\varepsilon'}\cos^2(\frac{\theta}{2})+\delta_{-\varepsilon'\varepsilon}\sin^2(\frac{\theta}{2})-\delta_{-\varepsilon'\varepsilon}v^2\sin^2(\frac{\theta}{2})\right) .$$

Averaging over incomong spin polarizations (unpolarized incoming current) and summing over outgoing spin polarizations (outgoing spin polarization is not measured) we obtain the same differential cross section formula as in the 1) case, as expected.
We notice that at larger scattering angles appears spin flipping during the scattering process in the case 2) when the spin orientation is in the motion plane.

As in Chap. Perturbation theory for the two-component Dirac equation, since the electron scatters in a Coulomb field we will have a spin-orbit interaction with a Larmor interaction energy given by

$\Delta H_L=-g_S\frac{\alpha}{2m^2r^3}\vec{L}\cdot\vec{S}$ with $\vec{L}=m\vec{x}\times\vec{v}$ the angular momentum, $g_S\approx 2$ the gyromagnetic ratio and $\alpha=\frac{eq}{4\pi}$, $\vec{x}$ the position vector of the electron pointing from the diffusion center, $\vec{v}=\frac{d\vec{x}}{dt}$, $r=\|\vec{x}\|$.

We have also a Thomas precession with an instantaneous rotation of the electron ret frame angular velocity

$\vec{\omega}_T=-\frac{1}{2}\vec{v}\times\vec{a}$ where $\vec{a}=\frac{d\vec{v}}{dt}$ is the acceleration of the electron.

In the classical Coulomb scattering we have an acceleration

$\vec{a}=\frac{1}{m}\frac{\alpha}{r^3}\vec{x}$ and so $\vec{\omega}_T=\frac{1}{2m^2}\frac{\alpha}{r^3}\vec{L}$, $\frac{d\vec{a}}{dt}=-\frac{3\alpha}{mr^5}(\vec{x}\cdot\vec{v})\vec{x}+\frac{1}{m}\frac{\alpha}{r^3}\vec{v}$.

For the Thomas precession contribution to the energy we have to consider the inertial forces acting on the spinning ball to which we approximate the electron in its rest frame, having a uniformly distributed mass with density ρ and spinning angular velocity ω.
In the electron rest frame R' , which has an instantaneous rotation of angular velocity ω_T (see Chap. Perturbation theory for the two-component Dirac equation) we have

the Euler inertial forces field with density $-\rho \frac{d\vec{\omega}_T}{ds'} \times \vec{x}'$ ($(s', \vec{x}')$ time-space coordinates in R') $\frac{dt}{ds'} = \gamma \approx 1$, $\gamma = \frac{1}{\sqrt{1-v^2}}$,

$\frac{d\vec{\omega}_T}{ds'} \approx \frac{d\vec{\omega}_T}{dt} = -\frac{1}{2}\vec{v} \times \left(\frac{3\alpha}{mr^5}(\vec{x}\cdot\vec{v})\vec{x}\right) = \frac{3\alpha}{2m^2r^5}(\vec{x}\cdot\vec{v})\vec{L}$.

In the cassical Coulomb scattering, when the electron is in the proximity of the diffusion centre, the distance *r* is near to its minimum and so $\vec{x}\cdot\vec{v} = \frac{1}{2}\frac{d}{dt}r^2 \approx 0$.

Therefore we can neglect the influence of the Euler force on the scattering process.

The centrifugal forces field is $F_{cf} = -\rho\vec{\omega}_T \times (\vec{\omega}_T \times \vec{x}')$ and has an energy $E_{cf} = \frac{4\pi\rho R^5}{15}\omega_T^2 = \frac{1}{5}R^2\omega_T^2 m$ where $R = \frac{3}{5}\frac{e^2}{4\pi m}$ is the estimated radius of the electron (see Chap. Perturbation theory for the two-component Dirac equation) .

Thus with $\frac{e^2}{4\pi} = \frac{\alpha}{Z}$ we have $E_{cf} = \frac{9}{500}\frac{\alpha^4}{Z^2 m^5 r^6}\vec{L}^2$.

Since in the classical Coulomb scattering $\vec{L}$ is constant we take the centrifugal forces energy contribution as a potential $W = W(r) \propto \frac{1}{r^6}$.

If we consider a scattering of electrons on impurities in a crystal lattice grid we can determine *Z* as the electric charge of one impurity node in the lattice grid which is the difference between the number of valence electrons of the impurity atom and the number of valence electrons of the majority atom of the lattice grid (*Z* can be positive or negative and has the same sign as α) .

The Coriolis inertial forces field is $F_{cor} = -2\rho(\vec{\omega}_T \times (\vec{\omega} \times \vec{x}'))$.

As we noticed in Chap. Perturbation theory for the two-component Dirac equation , for the spin angular momentum we must have $\vec{S} = \frac{2}{5}mR^2\vec{\omega}$ and so

$$F_{cor} = -\frac{5}{mR^2}\rho(\vec{\omega}_T \times (\vec{S} \times \vec{x}')) = \frac{125}{18}\frac{Z^2}{\alpha}\frac{\rho}{mr^3}((\vec{L}\cdot\vec{S})\vec{x}' - (\vec{L}\cdot\vec{x}')\vec{S}) .$$

If $\vec{L} \| \vec{S}$ (which is the situation in the 1) case), the Coriolis forces field is is conservative and we have a corresponding energy

$E_{cor} = \int_B \left(\int_{\Gamma(\vec{x}')} -2\rho(\vec{\omega}_T \times (\vec{\omega} \times \vec{x}''))d\vec{x}''\right) d^3\vec{x}'$ where B is the electron ball of radius R , $\Gamma(\vec{x}')$ is a path from the origin to $\vec{x}' \in B$.

Thus in the 1) case we have

$$E_{cor} = \frac{8\pi\rho}{15}R^5\vec{\omega}\cdot\vec{\omega}_T = \frac{\alpha}{2m^2r^3}\vec{L}\cdot\vec{S} .$$

In the 2) case we take the conservative part of the Coriolis force field which can be

$$\widetilde{F}_{cor}(\vec{x}\,')=\frac{125}{18}\frac{Z^2}{\alpha}\frac{\rho}{m r^3}((\vec{L}\cdot\vec{S})\vec{x}\,'-(L_i x'_i S_i)_{i=\overline{1,3}})$$ and

$$E_{cor}=\int_B(\int_{\Gamma(\vec{x}\,')}\widetilde{F}_{cor}(\vec{x}\,'')d\vec{x}\,'')d^3\vec{x}\,'=\frac{\alpha}{2m^2r^3}\vec{L}\cdot\vec{S}\ .$$

The dissipation generated by the non-conservative part of the Coriolis force field in the 2) case on a closed path Γ in the *R’* frame is according tho Stokes theorem proportional to the flux of $\vec{S}\times\vec{L}$ through the surface surrounded by Γ and since $\vec{L}$ is constant in the classical Coulomb scattering and as we will see the motion equations determined with the conservative part potential additional spin-orbit interaction energy are in the 2) case not dependendent on spin, to compensate this dissipation we will have the flipping of the spin angular momentum $\vec{S}$ during the scattering process in the 2) case.

Hence we have a spin-orbit interaction energy

$$\Delta H=\Delta H_L+W+E_{cor}=-\frac{(g_S-1)\alpha}{2m^2r^3}\vec{L}\cdot\vec{S}+\frac{9}{500}\frac{\alpha^4}{Z^2}\frac{1}{m^5r^6}\vec{L}^2\ .$$

As in the classical Coulomb scattering (see I. Ința, S. Dumitru Complemente de fizică) , we consider a shock parameter ρ (do not confuse with the density of the electron ball above) which is the distance between the initial motion line and the axis parallel to the incoming moment $\vec{p}$ with $\|\vec{p}\|=m v_\infty$ through the spinless heavy diffusion center so that we have $\vec{L}_\infty=-\rho m v_\infty e_3$ in the 1) case and $\vec{L}_\infty=-\rho m v_\infty e_1$ in the 2) case and in both cases we take $W=W(r)=\frac{9}{500}\frac{\alpha^4}{Z^2}\frac{1}{m^3r^6}\rho^2v_\infty^2\ ,\ r=\|\vec{x}\|$.

The work done by spin-orbit interaction forces to time moment *t* is

$\Delta H=W(r)-\frac{g_S-1}{2mr^3}\alpha(q\times\dot{q})\cdot\vec{S}-H_0$ where H_0 is a zero point energy and $q=\vec{x}$ are the position coordinates.

Therefore since $\Delta H=\int_0^t\left(\frac{\partial(\Delta H)}{\partial\dot{q}}\ddot{q}+\frac{\partial(\Delta H)}{\partial q}\dot{q}\right)dt$

and for $T=\frac{1}{2}m\dot{q}^2$ kinetic energy and $V=\frac{\alpha}{r}$ the Coulomb potential we must have

$\int_0^t\left(\frac{\partial T}{\partial\dot{q}}\ddot{q}-\frac{\partial T}{\partial q}\dot{q}\right)dt=\int_0^t-\frac{\partial V}{\partial q}\dot{q}\,dt-\Delta H$ on solutions with $\dot{q}(0)=\dot{q}(t)=0$

and integrating by parts we obtain

$$\int_0^t\left(\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}}\right)\dot{q}-\frac{\partial T}{\partial q}\dot{q}+\frac{\partial V}{\partial q}\dot{q}-\frac{d}{dt}\left(\frac{\partial(\Delta H)}{\partial\dot{q}}\right)\dot{q}+\frac{\partial(\Delta H)}{\partial q}\dot{q}\right)dt=0$$

we conclude that the Lagrangian of the system is

$L=L(q,\dot{q})=T-V-\Delta H=\frac{m\dot{q}^2}{2}-A(r)-m\frac{M}{r^3}(q\times\dot{q})\cdot\vec{S}$ where $A(r)=\frac{\alpha}{r}+W(r)$,

$M=-\frac{(g_S-1)\alpha}{2m^2}$, $r=\sqrt{q^2}$.

Introducing the generalized moment coordinates

$p=\frac{\partial L}{\partial\dot{q}}=m\dot{q}-m\frac{M}{r^3}\vec{S}\times q$ we have a Hamiltonian

$$H=p\dot{q}-L=\frac{p^2}{2m}+A(r)+\frac{M}{r^3}\vec{S}\cdot(q\times p)+\frac{1}{2}m\frac{M^2}{r^6}(\vec{S}\times q)^2$$

The Hamilton-Jacobi system is

$$\dot{q}=\frac{\partial H}{\partial p}=\frac{p}{m}+\frac{M}{r^3}\vec{S}\times q$$

$$\dot{p}=-\frac{\partial H}{\partial q}=-A'(r)\frac{q}{r}+3\frac{M}{r^5}(\vec{S}\cdot(q\times p))q+\frac{M}{r^3}(\vec{S}\times p)+3\frac{mM^2}{r^8}(\vec{S}\times q)^2q-$$
$$-\frac{mM^2}{r^6}(\vec{S}^2q-(\vec{S}\cdot q)\vec{S})$$

In the 1) case, we notice from the system that if $q(0),\dot{q}(0)$ are in the (e_1,e_2) plane , then the entire solution (q , p) remains in the (e_1 , e_2) plane. Thus we have to consider only the (q_1 , q_2 , p_1 , p_2) variables and in the 1) case we will have

$$q\times p\|\vec{S}\|e_3 \;,\; \vec{S}\cdot q=0 \;,\; \vec{S}\cdot p=0 \;,\; \dot{q}\times p+q\times\dot{p}=0 \;.$$

Hence in the 1) case $q\times p$ is conserved in time and also H is conserved in time.
Therefore we have $q\times p=\vec{\Lambda}=\Lambda e_3=-\rho m v_\infty e_3$,

$\frac{p^2}{2m}+A(r)+\frac{1}{8}\frac{M^2}{r^4}+\frac{M}{r^3}\vec{S}\cdot(q\times p)=E=\frac{1}{2}mv_\infty^2$ during the motion in the 1) case.

with $\vec{S}=\frac{1}{2}\varepsilon e_3$, $\varepsilon\in\{\pm1\}$.

From the HamiltonJacobi system we derive the motion equations

$$m\ddot{q}=\frac{\alpha}{r^3}q-\frac{W'(r)}{r}q+3\frac{M}{r^5}(\vec{S}\cdot(q\times p))q+\frac{3mM^2}{r^8}(\vec{S}^2q-(\vec{S}\cdot q)^2\vec{S})-$$
$$-\frac{3mM}{r^5}(q\cdot\dot{q})(\vec{S}\times q)+\frac{2mM}{r^3}(\vec{S}\times\dot{q})$$

In the 2) case we write the motion equations as $m\ddot{q}_i = F_i$, $i=\overline{1,3}$ and have the bounding $x=0$ choosing the generalized coordinates (q_2, q_3) with

$x=0$, $y=q_2$, $z=q_3$ and therefore the generalized forces are

$$Q_2 = F_1 \frac{\partial x}{\partial q_2} F_2 \frac{\partial y}{\partial q_2} + F_3 \frac{\partial z}{\partial q_2} = F_2$$

$$Q_3 = F_1 \frac{\partial x}{\partial q_3} + F_2 \frac{\partial y}{\partial q_3} + F_3 \frac{\partial z}{\partial q_3} = F_3 \ .$$

Thus in the 2) case we have

$$q=(0, q_2, q_3) \ , \ \vec{S} \times q = \frac{1}{2} \varepsilon(-q_2, 0, 0) \ ,$$

$$p = m\dot{q} - m\frac{M}{r^3}(\vec{S} \times q) = \left(\frac{\varepsilon m M}{2r^3} q_2, m\dot{q}_2, m\dot{q}_3\right) \ , \ (\vec{S} \times q)\cdot p = -\frac{mM}{4r^3} q_2^2 \ ,$$

$Q_2 = \left(\frac{\alpha}{r^3} - \frac{W'(r)}{r}\right) q_2$, $Q_3 = \left(\frac{\alpha}{r^3} - \frac{W'(r)}{r}\right) q_3$ and the motion equations are

$$m\ddot{y} = \left(\frac{\alpha}{r^3} - \frac{W'(r)}{r}\right) y$$

$$m\ddot{z} = \left(\frac{\alpha}{r^3} - \frac{W'(r)}{r}\right) z$$

We notice that in the 2) extreme case, with the spin angular momentum in the motion plane we have no spin dependence of the motion equations. However we have proved that in the 2) case spin flipping appears. Thus in an intermediate case with the spin angular momentum having an arbitrary direction and electrons confined to move in the flat conductor plane we expect differentiate scattering to the left or right of the charge current direction depending on spin polarization as we will see in the 1) case and also spin flipping during the scattering process.

Considering now the 1) case, we have

$$e_r = (\cos\theta, \sin\theta) \ , \ e_\theta = (-\sin\theta, \cos\theta) \ ,$$

$$q = r e_r \ , \ \dot{q} = \dot{r} e_r + r\dot{\theta}\ e_\theta \ , \ \ddot{q} = \ddot{r} e_r + 2\dot{r}\dot{\theta}\ e_\theta - r\dot{\theta}^2 e_r + r\ddot{\theta}\ e_\theta$$

$$\Lambda = mr\left(r\dot{\theta} + \frac{\varepsilon\alpha}{4m^2r^2}\right) \ , \ \frac{1}{2} m\dot{r}^2 = E - B(r) \ ,$$

$$B(r) = \frac{\alpha}{r} + W(r) - \frac{\alpha}{4}\frac{1}{m^2r^3}\varepsilon\Lambda + \frac{1}{32}\frac{\alpha^2}{m^3r^4} + \frac{\Lambda^2}{2mr^2} \qquad (1)$$

$$\left(\frac{d\theta}{dr}\right)^2 = \frac{\dot{\theta}^2}{\dot{r}^2} = \frac{\left(\Lambda - \frac{\varepsilon\alpha}{4mr}\right)^2}{2mr^4(E - B(r))}$$

For $\bar{\alpha}=\left|\frac{\alpha}{Z}\right|=\frac{e^2}{4\pi}\approx\frac{1}{137}$ we will require $\left|\frac{\Lambda}{\bar{\alpha}}\right|<1$ and also, since the electron is as a well defined particle on mass shell we must consider that r is greater than half the reduced Compton wavelenght (see Chap. Relativistic dynamics... Compton wavelenght) and so $mr>\frac{1}{2}$.

Thus $\left|\frac{\alpha}{4}\frac{1}{m^2r^3}\right|=2m|Z|O(\bar{\alpha}^2)$ and $\left|W(r)/\left(\frac{\alpha}{4}\frac{1}{m^2r^3}\Lambda\right)\right|=|Z|\frac{288}{500}O(\bar{\alpha}^4)$ and so in $O(\bar{\alpha}^4)$ approximation we can drop the $W(r)$ term in the $B(r)$ expression, leaving us with $B(r)=\frac{\alpha}{r}+\frac{\Lambda^2}{2mr^2}-\frac{\alpha}{4}\frac{1}{m^2r^3}\varepsilon\Lambda+\frac{1}{32}\frac{\alpha^2}{m^3r^4}$.

For $F(r)=E-B(r)$ the equation $F(r)=0$ has at least one positive root, since $F(\infty)=E>0$ and $F(0)=-\infty$. Considering the (1) relations it follows that from $t=-\infty$ when $\theta(-\infty)=\pi$ (we consider $\Lambda<0$) and $r(-\infty)=\infty$ to $t=0$ when $\theta(0)=\theta_0$ and $r(0)=r_m$, the function $r=r(t)$ is decreasing until at $t=0$ it reaches the minimum value r_m where r_m is the greatest positive root of $F(r)=0$ (the equation $F(r)=0$ is a quartic equation in $\frac{1}{r}$) . From $t=0$ to $t=\infty$ when $\theta(\infty)=\varphi$ where φ is the scattering angle and $r(\infty)=\infty$ the function $r=r(t)$ is increasing. $\dot{r}=\dot{r}(t)$ changes sign only at $t=0$ since r_m must be the greatest positive root of $F(r)=0$.

Therefore $\dot{r}(t)=-\sqrt{\frac{2}{m}F(r)}$ for $t<0$, $\dot{r}(t)=\sqrt{\frac{2}{m}F(r)}$ for $t>0$ and from (1) follows now $\theta(r(t))-\theta_0=(\operatorname{sign} t)\int_{r_m}^{r(t)}\frac{\Lambda-\frac{\varepsilon\alpha}{4rm}}{r^2\sqrt{2m}\sqrt{E-B(r)}}dr$

The unicity of solutions for the differential equations system in $r = r(t)$, $\theta = \theta(t)$ for $r(0)=r_m$, $\theta(0)=\theta_0$ leads to $\theta(t)-\theta_0=\theta_0-\theta(-t)$, $r(t)=r(-t)$ and so taking $\chi=-\int_{r_m}^{\infty}\frac{\Lambda-\frac{\varepsilon\alpha}{4rm}}{r^2\sqrt{2m}\sqrt{E-B(r)}}dr$ we will have $\varphi+2\chi=-\operatorname{sign}\Lambda\,\pi$

We have $E-B(r)=\frac{1}{E^3r^4}G(Er)$ where

$$G(x)=x^4-\alpha x^3-\frac{\Lambda^2}{2}\frac{E}{m}x^2+\frac{\varepsilon\alpha\Lambda}{4}\left(\frac{E}{m}\right)^2x-\frac{\alpha^2}{32}\left(\frac{E}{m}\right)^3 ,$$

$$\int_{r_m}^{\infty}\frac{\Lambda-\frac{\varepsilon\alpha}{4rm}}{r^2\sqrt{2m}\sqrt{E-B(r)}}dr=\int_{Er_m}^{\infty}\sqrt{\frac{E}{2m}}\frac{\Lambda-\frac{\varepsilon\alpha}{4x}\frac{E}{m}}{\sqrt{G(x)}}dx .$$

We consider $\alpha>0$. Taking $A=-\alpha$, $B=-\frac{\Lambda^2}{2}\frac{E}{m}$, $C=\frac{\varepsilon\alpha\Lambda}{4}\left(\frac{E}{m}\right)^2$,
$D=-\frac{\alpha^2}{32}\left(\frac{E}{m}\right)^3$, $a=-\frac{3}{8}A^2+B$, $b=\frac{A^3}{8}-AB+C$, $c=-\frac{3}{256}A^4+\frac{A^2B}{16}-\frac{AC}{4}+D$,
$p=-\frac{a^2}{12}-c$, $q=-\frac{a^3}{108}+\frac{ac}{3}-\frac{b^2}{8}$ after some calculus , for $\frac{E}{m}=O(\epsilon)$,
$\epsilon\to 0$, $\left|\frac{\Lambda}{\alpha}\right|<1$ we obtain
$b\neq 0$, $27q^2+4p^3>0$ and solutions of the equation $G(x)=0$ can be expressed as
$u(C,D)=-\frac{A}{4}+\frac{1}{2}\left(-\delta\sqrt{2y-a}\pm\sqrt{-2y-a+4\delta\sqrt{y^2-c}}\right)$ with $\delta\in\{\pm 1\}$,

$$y=\frac{a}{6}+w-\frac{p}{3w} \text{ , } w=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} \text{ and we have}$$

$u(C,D)=u(0,0)+O(\epsilon^2)$.
Thus in $O(v_\infty^4)$ approximation we can ignore the C,D terms in the expression of G so we can take $B(r)\approx\frac{\alpha}{r}+\frac{\Lambda^2}{2mr^2}$. (2)
Considering (2) and the case $\alpha>0$, $\left|\frac{\Lambda}{\alpha}\right|<1$, $v_\infty\ll 1$ we will have

$$m r_m v_\infty^2=\frac{\kappa^2 v_\infty^3}{-\alpha+\sqrt{\alpha^2+\kappa^2 v_\infty^3}}\approx 2\alpha \text{ where } \kappa^2=\rho^2 m^2 v_\infty \text{ .}$$

Therefore $\frac{\alpha}{4mr|\Lambda|}\leq\frac{\alpha}{4mr_m|\Lambda|}<\frac{1}{8}\frac{v_\infty}{|\rho|m}<\frac{1}{4}v_\infty\ll 1$ if $|\rho|$ is greater than half the reduced Compton wavelenght.
Hence $\operatorname{sign}(\theta-\theta_0)=\operatorname{sign}\Lambda$ for $t>0$ if $|\rho|$ exceeds half the reduced Compton wavelenght and $\left|\frac{\Lambda}{\alpha}\right|<1$ and $v_\infty\ll 1$.
If $\left|\frac{\Lambda}{\alpha}\right|<1$, $v_\infty<\alpha$, $v_\infty\ll 1$ we will have

$$\vec{v}^2=\dot{q}^2=\dot{r}^2+(r\dot{\theta})^2=\frac{2}{m}(E-B(r))+\left(\frac{\Lambda}{mr}-\frac{\alpha}{4(mr)^2}\right)^2<v_\infty^2+2\frac{\Lambda^2}{(mr_m)^2}+\frac{1}{8}\frac{\alpha^2}{(mr_m)^4}<$$

$$<v_\infty^2+\frac{1}{2}\left(\frac{\Lambda}{\alpha}\right)^2 v_\infty^4+\frac{1}{128}\left(\frac{v_\infty}{\alpha}\right)^2 v_\infty^6\ll 1.$$

We remain therefore, for the $\alpha > 0$ case, in the non-relativistic application domain during the entire motion of the electron.
Integrating with (2) expression for $B(r)$ we obtain

$$(\operatorname{sign} t)(\theta-\theta_0)=\int_{1/r}^{1/r_m}\frac{\Lambda-\frac{\varepsilon\alpha}{4m}\tau}{\sqrt{2m}\sqrt{E-\alpha\tau-\frac{\Lambda^2}{2m}\tau^2}}d\tau=$$

$$=(\operatorname{sign}\Lambda)\left(1+\frac{\varepsilon\alpha^2}{4\Lambda^3}\right)\left(\frac{\pi}{2}-\arcsin\frac{\frac{\Lambda^2}{r}+\alpha m}{\sqrt{2\Lambda^2 Em+\alpha^2 m^2}}\right)-\frac{\varepsilon\alpha v_\infty}{4\Lambda^2}\sqrt{1-\frac{2\alpha}{mrv_\infty^2}-\frac{\rho^2}{r^2}}\ .$$

$$\varphi=-(\operatorname{sign}\Lambda)\pi+2(\operatorname{sign}\Lambda)\left(1+\frac{\varepsilon\alpha^2}{4\Lambda^3}\right)\left(\frac{\pi}{2}-\arcsin\frac{\alpha m}{\sqrt{2\Lambda^2 Em+\alpha^2 m^2}}\right)-\frac{\varepsilon\alpha}{2\Lambda^2}v_\infty\ .$$

We consider further that $\alpha>0$, $v_\infty<\alpha$, $\frac{|\Lambda|}{\alpha}<1$ and it follows:

$$-(\operatorname{sign}\Lambda)\varphi=\pi-2\left(1+\frac{\varepsilon\alpha^2}{4\Lambda^3}\right)\arctan\frac{|\Lambda|v_\infty}{\alpha}+(\operatorname{sign}\Lambda)\frac{\varepsilon\alpha}{2\Lambda^2}v_\infty=\pi-2\left(1+\frac{\varepsilon\alpha^2}{4\Lambda^3}\right)\left(\frac{|\Lambda|}{\alpha}v_\infty+\right.$$

$$\left.-\frac{|\Lambda|^3}{\alpha^3}\frac{v_\infty^3}{3}+\frac{|\Lambda|^5}{\alpha^5}\frac{v_\infty^5}{5}-\ldots\right)+(\operatorname{sign}\Lambda)\frac{\varepsilon\alpha v_\infty}{2\Lambda^2}=\pi-2\arctan\frac{|\Lambda|v_\infty}{\alpha}+\frac{2}{3}\frac{\varepsilon v_\infty^3}{\alpha}\operatorname{sign}\Lambda+O(v_\infty^4).$$

$$\varphi\approx-(\operatorname{sign}\Lambda)\pi+2\arctan\frac{\Lambda v_\infty}{\alpha}-\frac{2}{3}\frac{\varepsilon v_\infty^3}{\alpha}\ .\quad (3)$$

We notice that in absence of spin effects ($\varepsilon = 0$) we obtain

$\cot^2\frac{\varphi}{2}=\frac{\Lambda^2 v_\infty^2}{\alpha^2}$ which is the dependence of the scattering angle φ on ρ from the classical Rutherford non-relativistic Coulomb scattering (see I. Ința , S. Dumitru , Complemente de fizică).

The relation (3) defines the dependence of the diffusion angle φ on the shock parameter ρ (since $\Lambda=-\rho m v_\infty$) when a flux of particles are scattered on the same diffusion center.

Let the number of particles having shock parameter in the interval $(\rho , \rho+d\rho)$, that are scattered in an unit of time be dN. These particles are scattered in the angular interval $(\varphi , \varphi+d\varphi)$. If j is the flux of incoming particles (the number of particles passing in an unit of time through a normal to motion direction unit surface element) we must have $dN=j2\pi|\rho d\rho|=j\pi|d\rho^2|$ and if $d\sigma$ is the differential cross section we have $dN=jd\sigma$. Thus $d\sigma=\pi|d\rho^2|$.

Since the conduction electrons in the flat conductor plate are restricted to move in direction $\vec{j}$ parallel to the (e_1,e_2) plane ($\vec{j}\|e_1$) we have to consider the number of electrons passing a normal section of height b and width dy parallel to the (e_2 , e_3) in an unit of time as $dN=jbdy$ and we can define the bidimensional flux $\tilde{\vec{j}}=\frac{dN}{dy}e_1=\vec{j}b$ and subsequently the bidimensional cross section $\tilde{\sigma}=\frac{1}{b}\sigma$ for the scattering in the (e_1,e_2) plane.

As we proved, if $A=(2\pi)^4 M\,\delta^4(p+k-p'-k')$ is a total scattering amplitude, then the differential cross section is

$$d\sigma=\frac{k_0 p_0}{(k_0+p_0)r}\frac{4m^2}{4k_0 p_0}\frac{1}{(2\pi)^2}\frac{d^3\vec{k}'}{2\omega(k')}\frac{d^3\vec{p}'}{2\omega(p')}|M|^2\delta^4(k+p-k'-p')=$$

$$=\frac{1}{(4\pi)^2}\frac{m^2}{(k_0+p_0)^2}|M|^2 d\Omega \text{ with } \omega(k')=\sqrt{\vec{k}'^2+\mu^2}\ ,\ \ \omega(p')=\sqrt{\vec{p}'^2+m^2}$$

Introducing cilindrical coordinates $(\bar{r},\bar{\theta},\bar{p})$, $\bar{\theta}\in(-\pi,\pi)$ with

$$x_1=\bar{r}\cos\bar{\theta}\ ,\ x_2=\bar{r}\sin\bar{\theta}\ ,\ x_3=\bar{p}\ ,\ r'=\sqrt{\bar{r}^2+\bar{p}^2}\ ,$$

$E=k_0+p_0=\sqrt{r'^2+\mu^2}+\sqrt{r'^2+m^2}$ leads to $r'=\frac{1}{2E}((E^2-(m+\mu)^2)(E^2-(\mu-m)^2))^{1/2}$.

Taking $I=\frac{d^3\vec{k}'}{2\omega(k')}\frac{d^3 p'}{2\omega(p')}\delta^4(p+k-p'-k')$

with $\vec{p}+\vec{k}=0$ in the mass-center frame we derive

$$I=\frac{1}{2\omega(k')}\frac{1}{2\omega(p')}r'^2\sin\psi\, d\bar{\theta}\, d\psi\,\delta(E-\sqrt{r'^2+\mu^2}-\sqrt{r'^2+m^2})\,dr'$$

where $\|\vec{k}'\|=\|\vec{p}'\|=r'$, $\frac{dr'}{dE}=\frac{\sqrt{r'^2+\mu^2}\sqrt{r'^2+m^2}}{Er'}$, $\psi\in(0,\pi)$.

Therefore the number of electrons scattered in an unit of time in directions defined by $\bar{\theta}\in(\theta,\theta+d\theta)$ is $dN=j\left(\int_0^\pi\int\frac{r'}{4E'}\sin\psi\,\delta(E'-E)|M|^2 dE'\,d\psi\right)\frac{m^2}{r(k_0+p_0)}\frac{1}{(2\pi)^2}d\bar{\theta}$

where we have $|M|^2=|M|^2(r,\bar{\theta},\varepsilon)$, $r'=r'(E')=r(E')$ defined above.

Hence $dN=j\frac{1}{2E^2}\frac{m^2}{(2\pi)^2}|M|^2 d\bar{\theta}$.

According to the interpretation of the bidimensional flux and cross section we have $dN=\tilde{j}\,d\tilde{\sigma}$ and so $dN=j\,d\sigma=\tilde{j}\,d\tilde{\sigma}=j\frac{1}{2E^2}\frac{m^2}{(2\pi)^2}|M|^2 d\bar{\theta}$

and since $\tilde{j}=jb$ we obtain $\frac{d\tilde{\sigma}}{d\bar{\theta}}=\frac{1}{b}\frac{1}{2E^2}\frac{m^2}{(2\pi)^2}|M|^2=\frac{2}{b}\frac{d\sigma}{d\Omega}$

Considering the flux of electrons that are scatteredby the same diffusion center when the motion of electrons is constrained to the (e_1 , e_2) plane we have the shock parameter $\bar{\rho}$ which is the distance between the trajectory of an incoming electron and the axial plane through the diffusion center, parallel to the flux vector $\vec{j}$ and perpendicular to the plate. Then the number of particles which have the shock parameter in the interval $(\bar{\rho},\bar{\rho}+d\bar{\rho})$ and are deflected in an unit time interval is $dN=\tilde{j}|d\bar{\rho}|$. The dN particles will be deflected in the scattering angle interval $(\bar{\theta},\bar{\theta}+d\bar{\theta})$ having $\bar{\rho}$ as a function of $\bar{\theta}$ and $d\tilde{\sigma}=\frac{dN}{\tilde{j}}=|d\bar{\rho}|$ (4) .

Thus $d\tilde{\sigma} = \frac{d\tilde{\sigma}}{d\bar{\theta}} d\bar{\theta} = \frac{2}{b} \frac{d\sigma}{d\Omega}(r,\theta) d\theta = |d\bar{\rho}|$.

If $\frac{d\sigma}{d\Omega}$ not depends on the angle ψ between $\vec{S}$ and the incoming moment (that is it depends only on incoming moment norm r)integrating with $d\Omega = \sin\psi d\bar{\theta} d\psi$ we obtain $\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 2\int \frac{d\sigma}{d\Omega} d\bar{\theta} = b\int \frac{d\tilde{\sigma}}{d\bar{\theta}} d\bar{\theta} = b\tilde{\sigma}$ and we verify $\tilde{\sigma} = \frac{\sigma}{b}$.

In the (e_1, e_2) plane we have $\vec{x}(r,\theta,\psi) = \vec{x}(\bar{r},\bar{\theta},\bar{p})$, $\psi = \frac{\pi}{2}$, $\bar{p} = 0$, $\theta = \bar{\theta}$, $r = \bar{r}$ where $\vec{x}(r,\theta,\psi) = (r\cos\theta, r\sin\theta\sin\psi, r\sin\theta\cos\psi)$ and therefore for the bidimensional approach $\theta = \bar{\theta}$ in the spherical coordinate argument of $\frac{d\sigma}{d\Omega}$.

In the motion plane , the tridimensional scattering angle can be identified with the plane scattering angle ($\theta = \bar{\theta} = \varphi$) and also the tridimensional shock parameter ρ can be identified with the bidimensional shock parameter $\bar{\rho}$.

The determination of $\frac{d\tilde{\sigma}}{d\bar{\theta}}$ is made in the supposition that the spin angular momentum is normal to the motion plane. We cannot extend the relation $\frac{d\tilde{\sigma}}{d\bar{\theta}} = \frac{2}{b}\frac{d\sigma}{d\Omega}$ to the entire tridiensional solid angle since this relation, as we proven may be valid for $\psi = \frac{\pi}{2}$ but the motion plane changes if we vary ψ and the spindirection is no more normal to the motion plane and as we have seen in the derivation for the 2) case equations, the motion becomes spatial and we have a dependence on ψ of the tridimensional cross section (r,θ,ψ) spherical coordinates. So we have $\frac{d\tilde{\sigma}}{d\bar{\theta}} = \frac{2}{b}\frac{d\sigma}{d\Omega}$ only in the $\psi = \frac{\pi}{2}$ plane.

For the situation we consider (a Copper with Iridium impurities plate) we have $Z \geq 1$ (Copper has one valence electron and Iridium has to nine valence electrons) and so $\alpha > 0$ and $|\bar{\rho}|$ must be considered smaller than half the minimum distance between impurity nodes in the plate crystal lattice grid (measured normal to charge current direction) which we denote $a/2$ where a is a lattice grid constant.

v_∞ is the drift velocity of the electrons determined by the charge current $e\vec{j}$.

Therefore we will have $\frac{|\Lambda|}{\alpha} < 1$, $v_\infty < \alpha$, $v_\infty \ll 1$ and since $\Lambda = -\bar{\rho} m v_\infty$, considering (3) we obtain

$$d\bar{\rho} = -\frac{\alpha}{2 m v_\infty^2} \frac{1}{\sin^2\left(\frac{\varphi}{2} + \frac{\varepsilon v_\infty^3}{3\alpha}\right)} d\varphi$$

with $\varphi = \bar{\theta}$ as the scattering angle in (e_1, e_2) plane,

$$\varphi \in (\varphi_+^1(\varepsilon), \varphi_+^2(\varepsilon)) \cup (\varphi_-^1(\varepsilon), \varphi_-^2(\varepsilon)) \quad \text{where}$$

$$\varphi_+^1(\varepsilon) = -\pi - \frac{2\varepsilon v_\infty^3}{3\alpha} \quad , \quad \varphi_+^2(\varepsilon) = -\pi + 2\arctan\left(\frac{a m v_\infty^2}{2\alpha}\right) - \frac{2\varepsilon v_\infty^3}{3\alpha} \quad ,$$

$$\varphi_-^1(\varepsilon) = -\varphi_+^2(-\varepsilon) \quad , \quad \varphi_-^2(\varepsilon) = -\varphi_+^1(-\varepsilon) \ .$$

From (4) follows now that the bidimensional differential cross section is

$$\frac{d\tilde{\sigma}_\varepsilon}{d\varphi}(\varphi) = \begin{cases} \dfrac{\alpha}{2 m v_\infty^2} \dfrac{1}{\sin^2\left(\dfrac{\varphi}{2} + \dfrac{\varepsilon v_\infty^3}{3\alpha}\right)} & \text{if } \varphi \in (\varphi_+^1(\varepsilon), \varphi_+^2(\varepsilon)) \cup (\varphi_-^1(\varepsilon), \varphi_-^2(\varepsilon)) \\ 0 & \text{else} \end{cases}$$

and we can verify that $\frac{d\tilde{\sigma}_\varepsilon}{d\varphi}(\varphi) = \frac{d\tilde{\sigma}_{-\varepsilon}}{d\varphi}(-\varphi)$ (5) .

The number of up-spin electrons that are deflected to the left of the incoming flux direction ($e_3 \times \vec{j}$ gives the left side direction of the flux vector direction) is

$$n_l(+1) = \tilde{j} \int_{\varphi_+^1(+1)}^{-\pi} \frac{d\tilde{\sigma}_+}{d\varphi} d\varphi + \tilde{j} \int_{\varphi_-^1(+1)}^{\varphi_-^2(+1)} \frac{d\tilde{\sigma}_+}{d\varphi} d\varphi = \tilde{j} \frac{\alpha}{m v_\infty^2} \left(\tan\left(\frac{v_\infty^3}{3\alpha}\right) + \frac{a m v_\infty^2}{2\alpha} \right)$$

and the number of up-spin electrons that are deflected to the right of the incoming flux direction in an unit of time is

$$n_r(+1) = \tilde{j} \int_{-\pi}^{\varphi_+^2(+1)} \frac{d\tilde{\sigma}_+}{d\varphi} d\varphi = \tilde{j} \frac{\alpha}{m v_\infty^2} \left(-\tan\left(\frac{v_\infty^3}{3\alpha}\right) + \frac{a m v_\infty^2}{2\alpha} \right)$$

We notice that $n_l(+1) > n_r(+1)$ and so we can see that up-spin electrons are deflected mostly to the left of the incoming flux direction or equivalent to the right of the charge current direction, since electrons carry negative charge and in the same way we conclude that down-spin electrons are deflected mostly to the left of the charge current direction (as we proved, in the considered 1) case, spin flipping during the scattering process is not allowed so up-spin will accumulate on the right edge of the plate with respect to charge current direction and down-spin will accumulate on the left edge of the plate).

In the bidimensional approach, we will have a mean free path of the electrons that are moving in the conductor plate plane given by $l = \frac{1}{\tilde{n}\tilde{\sigma}}$ (see Chap. Feynman amplitudes and lattice gauge theory), where $\tilde{n}$ is the areal concentration of impurity nodes in the plate and $\tilde{\sigma}$ is the total bidimensional cross section

$$\tilde{\sigma} = \int \frac{d\tilde{\sigma}}{d\varphi} d\varphi = a \ .$$

The number of electrons scattered in an unit of time by a diffusion center at angles in the interval $(\varphi, \varphi + d\varphi)$ and having spin polarization ε is (spin flipping is not allowed as we noticed) :

$d n_\varepsilon = \tilde{j} \dfrac{d\tilde{\sigma}_\varepsilon}{d\varphi}(\varphi) d\varphi$ which gives a particles bidimensional flux vector

$$\tilde{\vec{j}}_\varepsilon = \frac{d n_\varepsilon}{l\, d\varphi}(\varphi)(\cos\varphi, \sin\varphi)$$

Hence we will have a total bidimensional spin flux vector given by
$\tilde{\vec{j}}_s = \dfrac{1}{2}\int (\tilde{\vec{j}}_+(\varphi) - \tilde{\vec{j}}_-(\varphi)) d\varphi = \dfrac{\tilde{j}}{2l}\int \left(\dfrac{d\tilde{\sigma}_+}{d\varphi} - \dfrac{d\tilde{\sigma}_-}{d\varphi}\right)(\cos\varphi, \sin\varphi) d\varphi$.

From (5) follows that in the (e_1 , e_2) plane we have:

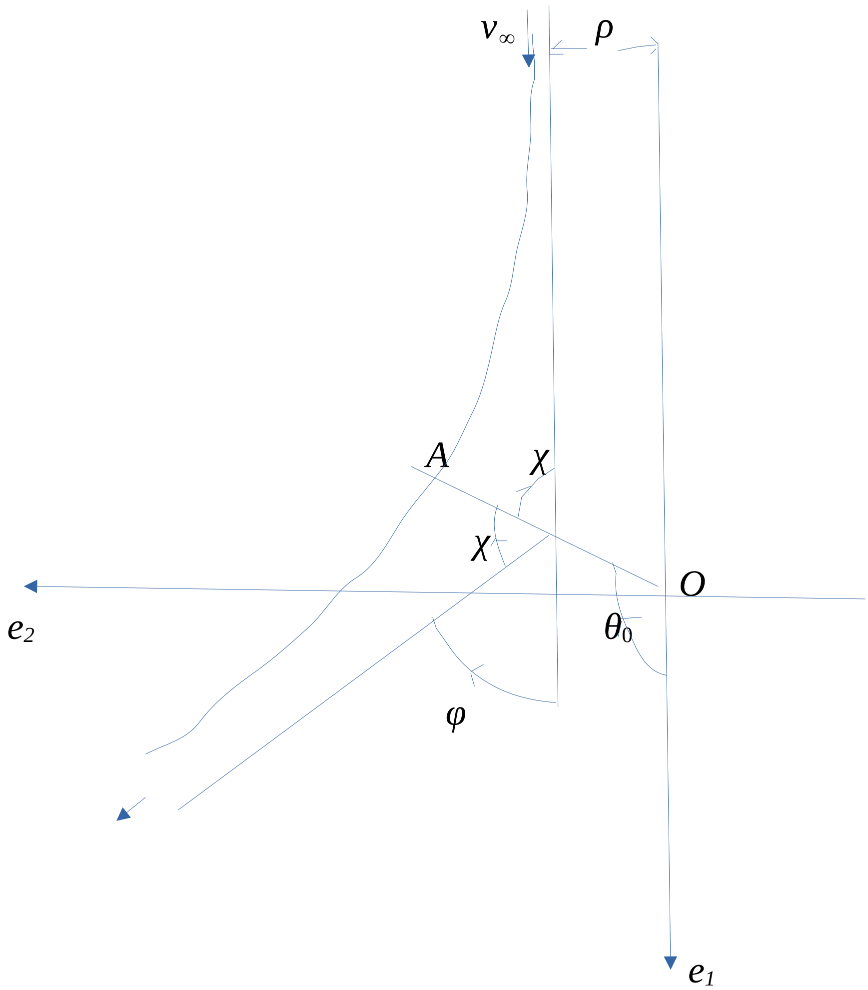
$$\tilde{\vec{j}}_s \cdot e_1 = 0 \quad \text{and} \quad \tilde{\vec{j}}_s \cdot e_2 = \frac{\tilde{j}}{l}\int \frac{d\tilde{\sigma}_+}{d\varphi} \sin\varphi d\varphi =$$

$$= \frac{\tilde{j}}{l}\frac{\alpha}{2mv_\infty^2} \int_{\pi - 2\arctan\left(\frac{amv_\infty^2}{2\alpha}\right)}^{\pi} \frac{1}{\sin^2(\frac{\varphi}{2})}\left(\sin\left(\varphi - \frac{2v_\infty^3}{3\alpha}\right) - \sin\left(\varphi + \frac{2v_\infty^3}{3\alpha}\right)\right) d\varphi =$$

$$= -\frac{\tilde{j}}{l}\frac{\alpha}{mv_\infty^2} \int_{\pi - 2\arctan\left(\frac{amv_\infty^2}{2\alpha}\right)}^{\pi} \sin\left(\frac{2v_\infty^3}{3\alpha}\right)\left(\frac{1}{\sin^2(\frac{\varphi}{2})} - 2\right) d\varphi =$$

$$= \left(-\tilde{j}\frac{a}{l} + 4\frac{\tilde{j}}{l}\frac{\alpha}{mv_\infty^2}\arctan\left(\frac{amv_\infty^2}{2\alpha}\right)\right)\sin\left(\frac{2v_\infty^3}{3\alpha}\right)$$

We verify that $\dfrac{amv_\infty}{2\alpha} < 1$ and $v_\infty \ll 1$ and so since $\lim\limits_{x\to 0}\dfrac{\arctan x}{x} = 1$ we derive that $\tilde{\vec{j}}_s \cdot e_2 > 0$. Therefore we have a spin current in the flat conductor plane that is normal to the charge current and is oriented to the right of the charge current direction (again we notice that the charge current has opposite orientation to the particles flux vector since electrons have negative charge). The apparition of the spin current pointing to the normal right direction of the charge current in a flat conductor sample is known as direct spin Hall effect.



For this figure we have $\Lambda=-\rho m v_\infty<0$, $\varphi+2\chi=\pi$, $\|OA\|=r_m$.

We assumed above conditions like $\frac{|\Lambda|}{\bar{\alpha}}<1$, $v_\infty<\bar{\alpha}$, $v_\infty\ll 1$ with $\bar{\alpha}=\frac{1}{137}$,
so we have to verify that the conditions (formulated in Planck units since we have already considered $\hbar=1$, $c=1$) :
$\frac{2\bar{\alpha}}{am}>v_\infty$, $v_\infty<\bar{\alpha}$, $v_\infty\ll 1$ (6) are experimentally available.

Time reversal not changes the spin-orbit coupling
$\vec{L}\cdot\vec{S}$ since $\vec{L},\vec{S}$ being angular momenta are both odd under time reversal.

Thus considering the time reversed inverse spin Hall effect, inverse spin Hall effect in which spin dependent skew scattering of spin carriers from an injected pure spin current on impurities in a crystal lattice grid, for example, generates a charge current giving a measurable inverse spin Hall effect signal ISHE of intensity I_C with a inverse spin Hall effect induced voltage U_{ISHE} , we will obtain a process equivalent to the direct spin Hall effect in which the drift velocity of electrons corresponding to the signal current intensity I_C is the scattering initial velocity v_∞ generating as described above the normal to its direction spin current.
We use the inverse spin Hall effect experiments data (from references [1], [2]) to illustrate a spin-orbit coupling influenced scattering of electrons on diffusion centers as presented above and verify that (6) assumptions are indeed experimentally available.
The experiments used a Copper with Iridium impurities plate of width $w = 10^{-7}$ m , height $b = 10^{-7}$ m . The Copper crystal lattice constant is $d = 3{,}6\ 10^{-10}$ m .
When V is the volume of the plate and c is the fraction of Iridium atoms from the total number of atoms in the sample we estimate the minimum distance between Iridium atoms in the lattice grid as a , having

$$c=\frac{\frac{V}{a^3}}{\frac{V}{d^3}+\frac{V}{a^3}} \text{ and so } \frac{V}{a^3}=\frac{V}{d^3}\frac{c}{1-c} \text{ , } a=d\sqrt[3]{\frac{1-c}{c}} \text{ .}$$

Copper has a density $\rho = 8{,}94\ 10^3$ kg / m^3 and a atomic mass $A = 63{,}5\ 10^{-3}$ kg / mol.
The concentration of atoms for Copper is $n = N_A\,\rho / A$ where $N_A = 6{,}023\ 10^{23}$ atoms / mol is the Avogadro number.
The measured ISHE voltages are in a range $U_{ISHE}\in(0,10^{-5})$ V (Volt) .
The inverse spin Hall effect resistances are in a range $|R_{ISHE}|\in(0,5\cdot10^{-5})\Omega(\text{Ohm})$ at $10\,K(\text{Kelvin})$ for a concentration $c=9\,\%$ and $|R_{ISHE}|\in(0,3\cdot10^{-5})\Omega$ for $c=6\,\%$.
The occurred flux of electrons $is\, j=\frac{I_C}{w\,b\,e}$ where $e=-1{,}6\cdot10^{-19}$C (Coulomb) is the electron charge and $I_C=\frac{U_{ISHE}}{R_{ISHE}}$ (the R_{ISHE} values are controlled through an applied magnetic field).
The Copper has one free electron per atom and so the concentration of conduction electrons is equal to n and the drift velocity of electrons that occurs is $v=\frac{j}{n}=\frac{U_{ISHE}}{R_{ISHE}}\frac{A}{N_A\rho}\frac{1}{w\,b\,e}$. Taking the maximal values for voltage and resistance we compute after dividing the value in m/s of v with the speed of light constant $3\cdot10^8$ m / s to obtain the value in Planck units :

$$v_\infty=3.4\cdot10^{-6} \text{ for } c=9\,\% \text{ and } v_\infty=5.66\cdot10^{-6} \text{ for } c=6\,\%.$$

We have the Planck lenght $l_P = 1.6 \cdot 10^{-35}$ m the Planck mass $m_P = 2.17 \cdot 10^{-8}$ kg and the electron mass $m = 9.1 \cdot 10^{-31}$ kg and so in Planck units we compute

$$am = \frac{d\sqrt[3]{\frac{1-c}{c}}\,m}{l_P m_P}\ ,\ am = 1.91 \cdot 10^3 \text{ for } c = 9\% \text{ and } am = 2.21 \cdot 10^3 \text{ for } c = 6\%.$$

We can verify that for both considered Iridium concentrations we have $\frac{2\bar{\alpha}}{am} > v_\infty$ and so the (6) assumptions are satisfied.

As we proved for $\frac{|\Lambda|}{\bar{\alpha}} < 1$, $v_\infty \ll 1$, $0 < \alpha$ we have the trajectory of the electron in the scattering process on a diffusion centre given by

$$\theta - \theta_0 = (\operatorname{sign}\Lambda)\left(1 + \frac{\varepsilon\alpha^2}{4\Lambda^3}\right)\left(\frac{\pi}{2} - \arcsin\frac{\frac{\Lambda}{r} + \alpha m}{\sqrt{2\Lambda^2 E + \alpha^2 m^2}}\right) - \frac{\varepsilon\alpha v_\infty}{4\Lambda^2}\sqrt{1 - \frac{\rho^2}{r^2} - \frac{2\alpha}{m r v_\infty^2}}$$

for $t > 0$ with $\theta(-t) - \theta_0 = \theta_0 - \theta(t)$ relation that can be written as

$$\theta - \theta_0 = (\operatorname{sign}\Lambda)\left(1 + \frac{\varepsilon\alpha^2}{4\Lambda^3}\right)\arcsin\frac{|\Lambda| v_\infty y}{\sqrt{\Lambda^2 v_\infty^2 + \alpha^2}} - \frac{\varepsilon\alpha v_\infty}{4\Lambda^2}\, y \quad \text{for } t > 0\ ,$$

where $y = \sqrt{1 - \frac{\Lambda^2 v_\infty^2}{z^2} - \frac{2\alpha}{z}}$, $z = m r v_\infty^2$, $z > z_m$, $z_m = \frac{\Lambda^2 v_\infty^2}{-\alpha + \sqrt{\Lambda^2 v_\infty^2 + \alpha^2}} \approx 2\alpha$

and we have $y = y(z)$, $y(z_m) = 0$, $y(\infty) = 1$, $y \in (0,1)$, $z \in (z_m, \infty)$.

The scattering angle is $\varphi = -\operatorname{sign}\Lambda\,\pi - 2\chi$ with $\theta = \theta(r)$, $\chi = (\theta_0 - \theta)(\infty)$

and so $\varphi = -\operatorname{sign}\Lambda\,\pi + 2\left(1 + \frac{\varepsilon\alpha^2}{4\Lambda^3}\right)\arctan\frac{\Lambda v_\infty}{\alpha} - \frac{\varepsilon\alpha v_\infty}{2\Lambda^2}$

having also $\theta_0 + \chi = -\operatorname{sign}\Lambda\,\pi$ as it follows from the figure.

We notice that if $(r, \theta(r))_r$ is the trajectory in polar coordinates for $(\operatorname{sign}\Lambda, \varepsilon) = (f, e) \in \{\pm 1\} \times \{\pm 1\}$ then $(r, -\theta(r))_r$ is the trajectory for $(\operatorname{sign}\Lambda, \varepsilon) = (-f, -e)$ with the same $|\Lambda|, v_\infty$.

We take $\operatorname{sign}\Lambda = -1$, $\Lambda = -\rho m v_\infty$, $\rho m > \frac{1}{2}$, $|\Lambda| < \bar{\alpha}$, $v_\infty \ll 1$.

Since for $t > 0$ we have $\theta - \theta_0 = \int_{r_m}^{r} \frac{\Lambda - \frac{\varepsilon\alpha}{4mr}}{r^2\sqrt{2m}\sqrt{E - B(r)}}\,dr$ and with the considered assumptions as we proved $\operatorname{sign}\left(\Lambda - \frac{\varepsilon\alpha}{4mr}\right) = \operatorname{sign}\Lambda$ we obtain that for $t > 0, \theta$ is a decreasing function of r and $\theta - \theta_0 < 0$, $\theta(r_m) = \theta_0$.

Therefore considering θ as a function of y we have that $\theta_0 - \theta$ is a increasing function of y .

For getting a relevant graphic representation of the trajectory in polar coordinates $r = r(\theta)$ we have $0 < \theta_0 - \theta < \chi = (\theta_0 - \theta)(1)$ (we consider further $\theta = \theta(y)$) .

We want an upper limitation for y (or equivalently for z or r) when $\theta_0-\theta\in(0,\chi-\delta)$ with $\delta>0$.
Because $\theta_0-\theta$ is increasing of y we derive that if $(\theta_0-\theta)(\bar{x})=\chi-\delta$ then for $y\in(0,\bar{x})$ we will have $(\theta_0-\theta)(y)\in(0,\chi-\delta)$.
Reminding that we have taken $\Lambda<0$, we have
$(\theta_0-\theta)'(y)=\left(1-\frac{\varepsilon\alpha^2}{4|\Lambda|^3}\right)\frac{1}{\sqrt{h^2-y^2}}+\frac{\varepsilon\alpha v_\infty}{4\Lambda^2}$ where $h=\sqrt{1+\frac{\alpha^2}{\Lambda^2 v_\infty^2}}$.
We have also $\frac{\alpha^2}{4|\Lambda|^3}=\frac{1}{4\alpha}\left(\frac{\alpha}{|\Lambda|}\right)^3>1$ and so $(\theta_0-\theta)'$ is increasing of y if $\varepsilon=-1$ and decreasing of y if $\varepsilon=1$ and obviously we have $(\theta_0-\theta)'>0$ in both cases.
Hence if $\varepsilon=-1$ we obtain $\delta=(\theta_0-\theta)(1)-(\theta_0-\theta)(\bar{x})<(\theta_0-\theta)'(1)(1-\bar{x})=$
$=\frac{|\Lambda|v_\infty}{\alpha}(1-\bar{x})$, $\bar{x}<1-\frac{\delta\alpha}{|\Lambda|v_\infty}$
In this case we have $\chi=\left(1+\frac{\alpha^2}{4|\Lambda|^3}\right)\arctan\frac{|\Lambda|v_\infty}{\alpha}-\frac{\alpha v_\infty}{4\Lambda^2}>$
$>\frac{|\Lambda|v_\infty}{\alpha}-\frac{1}{3}\frac{|\Lambda|^3 v_\infty^3}{\alpha^3}-\frac{v_\infty^3}{12\alpha}$. If we take v_∞ sufficiently small $v_\infty<\frac{1}{2}$ since already $\rho m>\frac{1}{2}$ and $|\Lambda|<\alpha$ we derive $\chi>\frac{1}{6}\frac{|\Lambda|v_\infty}{\alpha}$ and we can take $\delta=\left(\frac{\Lambda v_\infty}{\alpha}\right)^2\ll\chi$
obtaining $\bar{x}<1-\frac{|\Lambda|v_\infty}{\alpha}$
If $\varepsilon=1$ we have $\delta<(\theta_0-\theta)'(0)(1-\bar{x})=\left(\left(1-\frac{\alpha^2}{4|\Lambda|^3}\right)\frac{|\Lambda|v_\infty}{\sqrt{\Lambda^2 v_\infty^2+\alpha^2}}+\frac{\alpha v_\infty}{4\Lambda^2}\right)(1-\bar{x})$
If further $\alpha<\frac{1}{4}$ since $\rho m>\frac{1}{2}$, $|\Lambda|<\alpha$ we derive

$$\delta<\left(\left(1-\frac{\alpha^2}{4|\Lambda|^3}\right)\frac{|\Lambda|v_\infty}{\alpha+\frac{1}{2}\alpha v_\infty^2}+\frac{\alpha v_\infty}{4\Lambda^2}\right)(1-\bar{x})=$$

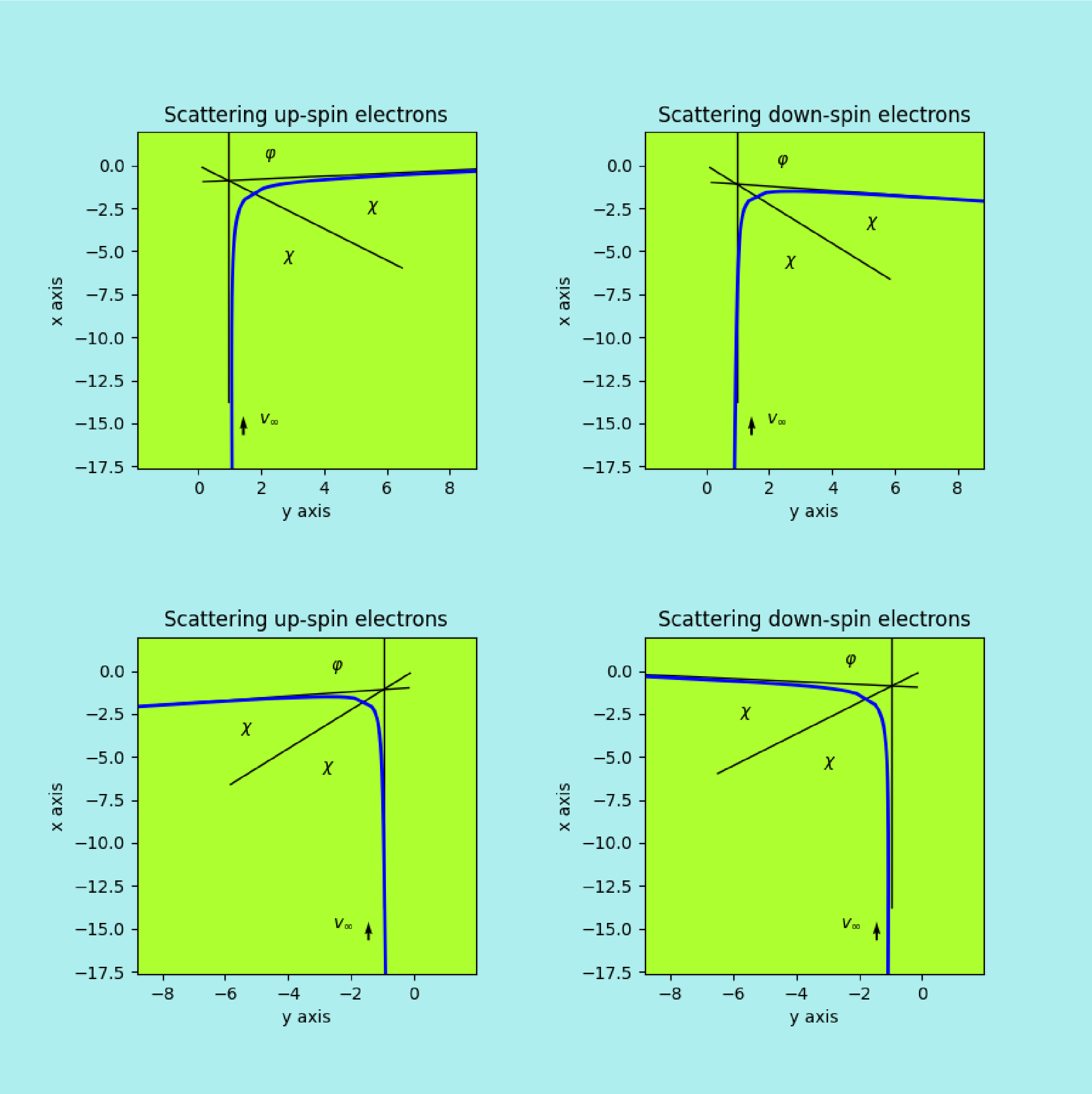
$$=\left(\frac{|\Lambda|v_\infty}{\alpha}+\frac{\alpha v_\infty}{8(\rho m)^2}\frac{1}{1+\frac{1}{2}v_\infty^2}\right)<\left(\frac{|\Lambda|v_\infty}{\alpha}+\frac{1}{2}\alpha v_\infty\right)(1-\bar{x})$$

Because $\alpha<\frac{1}{4}$ in this case we have $\chi=\left(1-\frac{\alpha^2}{4|\Lambda|^3}\right)\arctan\frac{|\Lambda|v_\infty}{\alpha}+\frac{\alpha v_\infty}{4\Lambda^2}>\frac{|\Lambda|v_\infty}{\alpha}$.
If $\frac{1}{2}\alpha v_\infty<\frac{|\Lambda|v_\infty}{\alpha}$ we can take $\delta=2\left(\frac{\Lambda v_\infty}{\alpha}\right)^2\ll\chi$ and if $\frac{|\Lambda|v_\infty}{\alpha}<\frac{1}{2}\alpha v_\infty$ we can take $\delta=\alpha v_\infty\frac{|\Lambda|v_\infty}{\alpha}\ll\chi$ in both situations
obtaining $\bar{x}<1-\frac{|\Lambda|v_\infty}{\alpha}$.

For $\bar{x}=\sqrt{1-\frac{\Lambda^2 v_\infty^2}{z^2}-\frac{2\alpha}{z}}$ we obtain the upper limitation for z as

$$\frac{1}{z}>\frac{1}{\alpha}\frac{-1+\sqrt{1+2\kappa^3-\kappa^4}}{\kappa^2}\approx\frac{1}{\alpha}\left(\kappa-\frac{1}{2}\kappa^2\right) \text{ where } \kappa=\frac{|\Lambda| v_\infty}{\alpha} .$$

The figure below shows the limit case $|\Lambda|=v_\infty=\alpha=0.98$.



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