Spin connection. Supergravity
Rarita-Schwinger action

Consider first the d-dimensional Minkowski space-time with signature $(-,+,+,+)$, $(\eta_{\mu\nu})_{\mu,\nu=\overline{0,d-1}}=$ diag $(-1,1,1,1)$, $d=2n$. The gamma matrices $(\gamma^\mu)_\mu$ satisfy $\gamma^\mu\in M_{2^n\times 2^n}(\mathbb{C})$, $\{\gamma^\mu , \gamma^\nu\}=2\eta^{\mu\nu}$.

(For $d=4$ in the Weyl representation

$\gamma^\mu =-i\begin{pmatrix}0 & \sigma^\mu\\ \tilde{\sigma}^\mu & 0\end{pmatrix}$ with $(\sigma^\mu)_\mu=(\boldsymbol{I} , (\sigma^j)_{j=\overline{1,3}})$, $(\tilde{\sigma}^\mu)_\mu=(\boldsymbol{I},(-\sigma^j)_{j=\overline{1,3}})$

$\sigma^1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\sigma^2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$, $\sigma^3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ the Pauli matrices).

We have $\gamma^{\mu+}=\gamma^0\gamma^\mu\gamma^0$.

The Lagrangian density for a mass m and q charged fermion field with $(A_\mu)_\mu$ gauge field is

$\mathcal{L}=-\overline{\psi}\gamma^\mu(\partial_\mu+iqA_\mu)\psi+m\overline{\psi}\psi$ where $\overline{\psi}=\psi^+ i\gamma^0$ leading to Dirac equation by variation upon $\delta\psi$ of the action $S=\int d^d x\mathcal{L}$

$d_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}-\frac{\partial\mathcal{L}}{\partial\psi}=0$ (Euler-Lagrange equations)

$\gamma^\mu\partial_\mu\psi+iqA_\mu\gamma^\mu\psi+m\psi=0$ (Dirac equation).

Taking $\psi_c= iC^{-T}\gamma^0\psi^*$ complex conjugating the Dirac equation we see that if $C^T\gamma^{\mu T}C^{-T}=-\gamma^\mu$ then $\psi_c^T C=\overline{\psi}$ and the conjugate field ψ_c satisfies the Dirac equation with opposite charge -q .

We notice that $C^T\gamma^{\mu T}C^{-T}=-\gamma^\mu$ leads to $-\gamma^{\mu T}=C^{-T}\gamma^\mu C^T=C^{-1}\gamma^\mu C$ for any μ and so $C^T=\lambda C$ with $\lambda\in\mathbb{C}$.

Also we must have $|\lambda|=1$ since $\det C=\det C^T\neq 0$.

For $n=1$ we take $\gamma^0=-i\sigma^1$, $\gamma^1=\sigma^{2.}$

For $n=2$ we have $\gamma^0=-i\sigma^1\otimes\boldsymbol{I}$, $\gamma^1=\sigma^2\otimes\sigma^1$, $\gamma^2=\sigma^2\otimes\sigma^2$, $\gamma^3=\sigma^2\otimes\sigma^3$

and we take also $\gamma_5=-i\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}\boldsymbol{I} & 0\\ 0 & -\boldsymbol{I}\end{pmatrix}$

For $m = 2n + 2$, $n > 1$ by induction we define

$$\gamma^k_{(m)}=\sigma^3\otimes\gamma^k_{(m-2)} \text{ for } k=\overline{0,\,2n-1}$$
$$\gamma^{2n}_{(m)}=\sigma^1\otimes\boldsymbol{I}_{2^n}\ ,\ \gamma^{2n+1}_{(m)}=\sigma^2\otimes\boldsymbol{I}_{2^n}\ ,\ \gamma_{2n+3}=-i^n\gamma^0\gamma^1\ldots\gamma^{2n+1} \text{ having}$$
$$\gamma_{2n+3}=\sigma^3\otimes\ldots\sigma^3\otimes\boldsymbol{I}$$

Under a Lorentz transformation $\Lambda=\exp(\omega^{\mu\nu}J_{\mu\nu})$ ($J_{\mu\nu}$ Lorentz group generators for $\mu,\nu=\overline{0,d-1}$) the spinor ψ transforms as $\psi\rightarrow S(\Lambda)\psi$ with $S(\Lambda)=\exp(\frac{1}{2}\omega^{\mu\nu}\sigma_{\mu\nu})$, $\sigma_{\mu\nu}=\frac{1}{2}[\gamma_\mu,\gamma_\nu]$
(rising and lowering Lorentz indices with $B_\mu=\eta_{\mu\nu}B^\nu$, $\eta_{\mu\nu}=\eta^{\mu\nu}$ and Einstein summation convention for repeating indices).
We take $C\in M_{2^n\times2^n}(\mathbb{C})$ such that $C\gamma^{\mu T}C^{-1}=-\gamma^\mu$ and we notice that if $\overline{\psi}=\psi^TC$ then for $\psi'=\gamma_\mu\gamma_\nu\psi$, $\mu\neq\nu$ we have $\overline{\psi}'=\psi'^TC$ and we conclude that the spinors satisfying $\overline{\psi}=\psi^TC$ form a irreducible spinor representation of the d-dimensional restricted Lorentz group called the Majorana representation.
For $d=2$ we take $C=i\sigma^2$ and for $d=2n$, $n\geq2$ we can take
$C=\varepsilon\gamma^1\gamma^3\ldots\gamma^{2n-1}$ if $n\equiv1$ (mod 2) and we have $C^T=(-1)^{\frac{n(n+1)}{2}}=C^{-1}$
$C=\varepsilon\gamma^0\gamma^2\ldots\gamma^{2n-2}$ if $n\equiv0$ (mod 2) and we have $C^T=(-1)^{\frac{n(n-1)}{2}}=C^{-1}$
where $\varepsilon\in\mathbb{C}$, $|\varepsilon|=1$ is chosen such that C is a real matrix.
For $n=2$ we take $C=-i\gamma^0\gamma^2=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}$ and $\overline{\psi}=\psi^TC$ reads into $\psi^*=i\gamma^0C^T\psi$ equivalent to $\psi=\psi_c$ so that any 4-dimensional Majorana spinors are those of the form $\psi^T=(a,b,b^*,-a^*)$
with $a,b\in\mathbb{C}$.

In the general dimension case Majorana spinors are also defined by $\overline{\psi}=\psi^TC$, $\psi^*=i\gamma^0C^T\psi$ which means that $\Re\psi$ and $\Im\psi$ are eigenvectors of $i\gamma^0C^T$ corresponding to the eigenvalue $+1$ and respective -1 and so the Majorana spinors space is a 2^n-dimensional real vector space spanned by the set $\{\Re\psi,\Im\psi|\psi$ - Majorana spinor $\}$

which set also spans the entire 2^n -dimensional spinor space as a complex vector space, $\Re\psi$, $\Im\psi$ being independent vectors, since they are eigenvectors to different eigenvalues of the same operator that has only +1 and and -1 eigenvalues with 2^{n-1} multiplicity.
Thus the Lagrangian density for massless Majorana spinor fields can be taken as $\mathscr{L}=-\frac{1}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi=-\frac{1}{2}\psi^T\gamma^\mu\partial_\mu\psi$ which is as expected real if ψ is a Majorana spinor and $C=-C^T$ as it happens if $d=2,4,10$ and so since in those cases $C\gamma^\mu$ are symmetric matrices, anticommuting Majorana spinors are allowed by this Lagrangian density because integrating by parts for vanishing at infinity fields we have
$\int d^d x\psi^T C\gamma^\mu\partial_\mu\psi=-\int d^d x\partial_\mu\psi^T C\gamma^\mu\psi=\int d^d x\psi^T C\gamma^\mu\partial_\mu\psi$
(anticommuting fields and symmetric matrix)
For $d=10$ we take $C=-i\gamma^1\gamma^3\gamma^5\gamma^7\gamma^9$ and we notice that $iC^{-T}\gamma^0=\sigma^1\otimes\sigma^2\otimes\sigma^1\otimes\sigma^1\otimes\sigma^2$ commutes with γ_{11} and they are both real matrices. So for $d=10$ (as for $d=2$) we can have Majorana-Weyl spinors that are in a irreducible representation for which
$\psi=\psi_c$ and $\gamma_{11}\psi=\varepsilon\psi$ are simultaneously satisfied.
(where ε is +1 or -1).
The Majorana-Weyl spinor space, if exists has real dimension 2^{n-1}.
For $d=4$, ψ and ψ_c have opposite chiralities and we cannot have Majorana-Weyl spinors. We have the irreducible representations of Majorana spinors satisfying $\psi=\psi_c$ with real dimension $2^n=4$ and of Weyl spinors satisfying $\gamma_5\psi=\varepsilon\psi$ with complex dimension $2^{n-1}=2$, which is also a real 4 dimension.

Gravity is defined by the metric $(g_{\mu\nu})_{\mu\ ,\ \nu}$ that defines the Levi-Civita connection $(\Gamma^\mu_{\nu\rho})_{\mu,(\nu,\rho)}$ which plays the role of a gauge field of gravity with respect to the general coordinates transformation invariance of the action, having the Riemann tensor $(R^\mu_{\nu\rho\sigma}(\Gamma))_{\mu,\nu,\rho,\sigma}$ as field strenght.
Any curved space is locally flat, if we look at a scale much smaller than the scale of the curvature. That means that locally we have the Lorentz invariance of special relativity. So we can define the vielbein field
$(e^a_\mu)_{\mu\ ,\ a}$ with $\mu=\overline{0,d-1}$, $a=\overline{0,d-1}$ such that
$g_{\mu\nu}(x)=e^a_\mu(x)e^b_\nu(x)\eta_{ab}$.

In $e_\mu^a(x)$, μ is a 'curved' index , acted upon by a general coordinate transformation so that $(e_\mu^a)_\mu$ is a covariant vector of general coordinate transformations, like a gauge field and *a* is a newly introduced ‘flat’ index acted upon by a local Lorentz invariance. That is around each point *x* we define a small flat neighborhood called ‘tangent space’ and *a* is a tensor index living in that local Minkowski space, acted upon by Lorentz transformations. The Lorentz transformation is local, since the tangent space on which it acts changes at each point on the curved manifold.

$(g_{\mu\nu})_{\mu,\nu}$ has $\frac{d(d+1)}{2}$ components. On e_μ^a we act with another local symmetry, not present in the metric, local Lorentz invariance, given as

$e_\mu^a \rightarrow \Lambda_b^a e_\mu^b$, $\Lambda=\exp(\lambda^{ab} J_{ab})\in SO^+(d-1,1)$, $\lambda^{ab}=-\lambda^{ba}=\lambda^{ab}(x)$

so we can set $\frac{d(d-1)}{2}$ (the Lie group dimension of $SO^+(d-1,1)$) components of $(e_\mu^a)_{\mu,a}$ to zero and so in fact the vielbein has $d^2-\frac{d(d-1)}{2}=\frac{d(d+1)}{2}$ independent components,the same as the metric.

On both the metric and the vielbein we have general coordinate transforms $x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu$ with an infitesimal form $\delta x^\mu=\xi^\mu(x)$.

The infinitesimal coordinate transform (Einstein transformation), acting on the metric gives $g_{\mu\nu} \rightarrow g'_{\mu\nu}=g_{\mu\nu}+\delta_\xi g_{\mu\nu}(x)$

$$\delta_\xi g_{\mu\nu}(x)=(\xi^\rho \partial_\rho) g_{\mu\nu}+(\partial_\mu \xi^\rho) g_{\rho\nu}+(\partial_\nu \xi^\rho) g_{\rho\mu}$$

where the first term corresponds to a translation $x^\mu \rightarrow x^\mu+\xi^\mu$ (in special we have a global parameter ξ^μ but now we have a local $\xi^\mu(x)$ that is the local version of the global μ translations in special relativity and the other terms correspond to the tensor coefficients transformations under the general coordinates transform.

On the vielbein, the infinitesimal Einstein transformation gives

$e_\mu^a \rightarrow e_\mu^a+\delta_\xi e_\mu^a(x)$, $\delta_\xi e_\mu^a(x)=(\xi^\rho \partial_\rho) e_\mu^a+(\partial_\mu \xi^\rho) e_\rho^a$.

On the other hand , the local infinitesimal Lorentz transformation is

$\Lambda_b^a=\delta_b^a+2\lambda_b^a$, $\lambda_b^a=\lambda_b^a(x)=\eta_{bc}\lambda^{ac}$, $\lambda^{ab}=-\lambda^{ba}$

$e_\mu^a \rightarrow e_\mu^a+\delta_{lL} e_\mu^a(x)$, $\delta_{lL} e_\mu^a(x)=2\lambda_c^a(x) e_\mu^c(x)$.

For a general covariant vector field $(A_\mu(x))_\mu$ we have a local Lorentz vector $(A_a(x))_a$ such that $A_\mu=e_\mu^a A_a$.

We define a spin connection $\left(\omega_\mu^{ab}(x)\right)_{\mu,a,b}$ derivation rule D_μ such that
$D_\mu(AB)=(D_\mu A)B+A(D_\mu B)$ (product rule)
$D_\mu A=\partial_\mu A$ for A a scalar field,
$D_\mu A^a=\partial_\mu A^a+\omega_\mu^{ab}A_b$ (for $A_b=\eta_{bc}A^c$, $(A^a)_a$ a local Lorentz vector),
$D_\mu\psi=\partial_\mu\psi+\frac{1}{4}\omega_\mu^{ab}\sigma_{ab}\psi$ with $\sigma_{ab}=\frac{1}{2}[\gamma_a,\gamma_b]$ for ψ a local spinor,
$D_\mu A_\nu=\partial_\mu A_\nu-\Gamma_{\mu\nu}^\lambda A_\lambda$ for $(A_\lambda)_\lambda$ a covariant vector.
Thus $D_\nu e_\mu^a=\partial_\nu e_\mu^a+\omega_{\nu b}^a e_\nu^b-\Gamma_{\nu\mu}^\lambda e_\lambda^a$.
Requiring that $e_\mu^a e_b^\mu=\delta_b^a$, $e_a^\mu e_\lambda^a=\delta_\lambda^\mu$ since $D_\nu(\delta_\lambda^\mu)=0$, the product rule leads to $D_\nu e_a^\mu=\partial_\nu e_a^\mu-\omega_{\nu a}^b e_b^\mu+\Gamma_{\nu\lambda}^\mu e_a^\lambda$.
For a multiindexed $(A_{a1a2...ak})_{a1,a2,...ak}$ we define
$A_{[a1a2...ak]}=\frac{1}{k!}\sum_{\sigma\in S_k}\varepsilon(\sigma)A_{a\sigma1a\sigma2...a\sigma k}$ where S_k is the set of $(1,2,...k)$ permutations and $\varepsilon(\sigma)$ is the signature of the σ permutation.
Requiring that the spin connection is torsion free, that is $D_{[\nu}e_{\mu]}^a=0$ and antisymmetric, that is $\omega_\mu^{ab}=-\omega_\mu^{ba}$ we obtain a system in the $\frac{d^2(d-1)}{2}$ unknowns $\omega_\mu^{[ab]}=\omega_\mu^{ab}$ of $\frac{d^2(d-1)}{2}$ independent equations given by
$$\omega_{[\nu}^{ab}e_{\mu]b}=-\partial_{[\nu}e_{\mu]}^a \quad \text{for} \quad a,\mu,\nu=\overline{0,d-1}. \qquad (1)$$
The homogeneous system in the antisymmetric spin connection coefficients corresponding to (1) leads to
$e_{\lambda a}\omega_\mu^{ab}e_{\nu b}=e_{\lambda a}\omega_\nu^{ab}e_{\mu b}$ for $\lambda,\mu,\nu=\overline{0,d-1}$.
Hence taking $\Omega_{\lambda\mu\nu}=e_{\lambda a}\omega_\mu^{ab}e_{\nu b}$ since $\omega_\mu^{ab}=-\omega_\mu^{ba}$ we have
$\Omega_{\lambda\mu\nu}=-\Omega_{\nu\mu\lambda}=-\Omega_{\mu\nu\lambda}=-\Omega_{\mu\lambda\nu}=-\Omega_{\lambda\mu\nu}$ and so $\Omega_{\lambda\mu\nu}=0$,
leading to $\omega_\mu^{ab}=0$ and therefore (1) has an unique antisymmetric in a,b solution which is
$$\omega_\mu^{ab}=\frac{1}{2}e^{\nu a}(\partial_\mu e_\nu^b-\partial_\nu e_\mu^b)+\frac{1}{2}e^{\nu b}(\partial_\nu e_\mu^a-\partial_\mu e_\nu^a)+\frac{1}{2}e^{\rho a}e^{\sigma b}(\partial_\sigma e_{\rho c}-\partial_\rho e_{\sigma c})e_\mu^c$$
and satisfies $\omega_\mu^{ab}=e^{\nu b}\Gamma_{\mu\nu}^\rho e_\rho^a-e^{\nu b}\partial_\mu e_\nu^a$,
$D_\mu e_\nu^a=0$, $D_\mu\eta^{ab}=0$, $D_\mu g_{\nu\lambda}=0$.

Let $R^{ab}_{\mu\nu}(\omega)=\partial_\nu\omega^{ab}_\mu-\partial_\mu\omega^{ab}_\nu-\omega^{ap}_\mu\omega^{qb}_\nu\eta_{pq}+\omega^{ap}_\nu\omega^{qb}_\mu\eta_{pq}$.
We have

$e^\mu_a e_{\nu b}\omega^{ab}_\rho=\Gamma^\mu_{\rho\nu}-(\partial_\rho e^a_\nu)e^\mu_a$ and applying D_σ to this:

$$e^\mu_a e^b_\nu(\partial_\sigma\omega^a_{\rho b}-\Gamma^\lambda_{\sigma\rho}\omega^a_{\lambda b}+\omega^a_{\sigma c}\omega^c_{\rho b}-\omega^c_{\sigma b}\omega^a_{\rho c})=$$
$$=\partial_\sigma\Gamma^\mu_{\rho\nu}+\Gamma^\mu_{\lambda\sigma}\Gamma^\lambda_{\rho\nu}-\Gamma^\lambda_{\rho\sigma}\Gamma^\mu_{\lambda\nu}-\Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\lambda\rho}-(\partial_\rho\partial_\sigma e^a_\nu)e^\mu_a-\partial_\rho e^a_\nu\partial_\sigma e^\mu_a-$$
$$-\Gamma^\mu_{\lambda\sigma}(\partial_\rho e^a_\nu)e^\lambda_a+\Gamma^\lambda_{\rho\sigma}(\partial_\lambda e^a_\nu)e^\mu_a+\Gamma^\lambda_{\nu\sigma}(\partial_\rho e^a_\lambda)e^\mu_a$$

which by antisymmetrization with respect to ρ , σ leads to
$e^\mu_a e_{\nu b}R^{ab}_{\rho\sigma}(\omega)=R^\mu_{\nu\rho\sigma}(\Gamma)$ where $(R^\mu_{\nu\rho\sigma})_{\mu,\nu,\rho,\sigma}$ is the Riemann curvature tensor $R^\mu_{\nu\rho\sigma}=\Gamma^\mu_{\nu\rho,\sigma}-\Gamma^\mu_{\nu\sigma,\rho}+\Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma}-\Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\lambda\rho}$. The Ricci tensor is $R_{\nu\sigma}=-R^\mu_{\nu\mu\sigma}=e^\mu_a e_{\nu b}R^{ab}_{\sigma\mu}$ and the Ricci scalar is $R=g^{\nu\sigma}R_{\nu\sigma}=e^\mu_a e^\sigma_b R^{ba}_{\mu\sigma}$.

The Hilbert-Einstein action $S_{EH}=\frac{1}{16\pi G}\int d^d x\sqrt{-g}\,R$ can be therefore written as $S_{EH}(e,\omega)=\frac{1}{2k_N^2}\int d^d x(\det e)e^\mu_a e^\sigma_b R^{ba}_{\mu\sigma}(\omega)$ (2) .
(with $k_N=\sqrt{8\pi G}$ a gravitational Newton constant in d dimensions).
We notice that the (2) form of the action is orientation dependent.
An orientation independent form can be obtained by multiplication with $\mathrm{sign}(\det e)$ since in fact $\sqrt{-g}=|\det e|$.

For $d=4$ we denote everywhere $\epsilon_{abcd}=\epsilon^{abcd}$ for any $a,b,c,d\in\{0,1,2,3\}$ the signature of the permutation $\begin{pmatrix}0&1&2&3\\a&b&c&d\end{pmatrix}$
and we have $2(\det e)(e^\mu_a e^\sigma_b-e^\sigma_a e^\mu_b)=\epsilon_{cdab}\epsilon^{\rho\nu\mu\sigma}e^c_\rho e^d_\nu$ and so
$S_{EH}(e,\omega)=\frac{1}{8k_N^2}\int d^4 x\,\epsilon_{abcd}\epsilon^{\rho\nu\mu\sigma}e^c_\rho e^d_\nu R^{ba}_{\mu\sigma}(\omega)$ (3) .

If fermions are involved, the action we will consider is of the form $S(e,\omega,\psi)=S_{EH}(e,\omega)+S_F(e,\omega,\psi)$ (ψ standing for fermionic fields) where considering the derivative of the form

$D_\mu\psi=\partial_\mu\psi+\frac{1}{8}\omega^{ab}_\mu[\gamma_a,\gamma_b]\psi$ we have

$S_F(e,\omega,\psi)=\int d^d x L_F(e,\omega,\psi,\partial\psi)$ with L_F linear in ω .
Also $S_{EH}(e,\omega)=\int d^d x L_{EH}(e,\omega,\partial\omega)$ with

$L_{EH}=Q_{EH}(e,\partial\omega)+P_{EH}(e,\omega)$ where Q_{EH} is linear in $\partial\omega$ and P_{EH} is quadratic in ω .

Now we can have a first order formalism for the theory. In the first order formalism, all fields *e*, *ω*, *ψ* are independent and we have $\delta S=\frac{\delta S}{\delta e}\delta e+\frac{\delta S}{\delta\omega}\delta\omega+\frac{\delta S}{\delta\psi}\delta\psi$ giving Euler-Lagrange equations for each field e,ω,ψ.

In the second order formalism we solve the Euler-Lagrange equations for the ω field, giving $\omega=\omega(e,\psi)$ corresponding to $\frac{\delta S}{\delta\omega}(\omega(e,\psi),e,\psi)=0$ and then we substitute ω by $\omega(e,\psi)$ in the action expression, leading to an action expression $\hat{S}(e,\psi)=S(\omega(e,\psi),e,\psi)$ and we will have Euler-Lagrange equations for e and ψ.

We notice that, since $\frac{\delta S}{\delta\omega}(\omega(e,\psi),e,\psi)=0$ and

$\delta\hat{S}=\frac{\delta S}{\delta e}\delta e+\frac{\delta S}{\delta\psi}\delta\psi+\frac{\delta S}{\delta\omega}\left(\frac{\partial\omega}{\partial e}\delta e+\frac{\partial\omega}{\partial(\partial e)}\partial(\delta e)+\frac{\partial\omega}{\partial\psi}\delta\psi\right)$ the Euler-Lagrange equations for *e* and *ψ* will be equivalent to $\frac{\delta S}{\delta e}(\omega(e,\psi),e,\psi)=0$ and $\frac{\delta S}{\delta\psi}(\omega(e,\psi),e,\psi)=0$.

Because of the form of the Lagrangian density, the Euler Lagrange equations $d_\mu\frac{\partial L}{\partial(\partial_\mu\omega)}-\frac{\partial L}{\partial\omega}=0$ lead to an expression $\omega(e,\psi)=\overline{\omega}(e)+\hat{\omega}(e,\psi)$ where $\omega(e)$ satisfies $\frac{\partial P_{EH}}{\partial\omega}(e,\overline{\omega}(e))=d_\lambda\frac{\partial Q_{EH}}{\partial(\partial_\lambda\omega)}(e)$ and $\hat{\omega}(e,\psi)$ satisfies $\frac{\partial P_{EH}}{\partial\omega}(e,\hat{\omega}(e,\psi))=-\frac{\partial L_F}{\partial\omega}(e,\hat{\omega}(e,\psi),\psi)$.

After some calculus we obtain that for determining the spin connection coefficients $\overline{\omega}_\mu^{ab}(e)$ and $\hat{\omega}_\mu^{ab}(e,\psi)$ we will have equations

$W_{\lambda\mu\eta}=\delta_{\lambda\mu}E_{\eta\delta\delta}+\delta_{\eta\mu}E_{\delta\delta\lambda}-g_{\lambda\varepsilon}g^{\rho\mu}E_{\eta\varepsilon\rho}-g_{\eta\varepsilon}g^{\rho\mu}E_{\rho\varepsilon\lambda}$ (4) , where $W_{\lambda\mu\eta}=e_\lambda^a\Omega_{\mu ab}e_\eta^b$, $E_{\eta\varepsilon\rho}=e_{\eta c}\omega^{\varepsilon cp}e_{\rho p}$

$$\Omega_{\mu ab}=\overline{\Omega}_{\mu ab}=\frac{2k_N^2}{\det e}d_\lambda\frac{\partial Q_{EH}}{\partial(\partial_\lambda\omega_\mu^{ab})}(e,\overline{\omega}) \text{ for } \overline{\omega} \text{ and}$$

$$\Omega_{\mu ab}=\widehat{\Omega}_{\mu ab}=-\frac{2k_N^2}{\det e}\frac{\partial L_F}{\partial\omega_\mu^{ab}}(e,\widehat{\omega},\psi) \text{ for } \widehat{\omega} \text{ .}$$

We expect $\Omega_{\mu ab}$ to be antisymmetric in a,b and therefore $W_{\lambda\mu\eta}$ antisymmetric in λ,η . Supposing the solution ω_μ^{ab} of the (4) system is antisymmetric in a,b and therefore we seek for $E_{\eta\varepsilon\rho}$ antisymmetric in η,ρ the system (4) reduces to a system of $\frac{d^2(d-1)}{2}$ for $\frac{d^2(d-1)}{2}$ variables $(E_{\eta\varepsilon\rho}=-E_{\rho\varepsilon\eta})_{\eta,\varepsilon,\rho}$ and to show that the system (4) has an unique antisymmetric solution we will show that if $E_{\rho\varepsilon\eta}=-E_{\eta\varepsilon\rho}$ and

$\delta_{\mu\lambda}E_{\eta\delta\delta}+\delta_{\mu\eta}E_{\delta\delta\lambda}-g_{\lambda\varepsilon}g^{\rho\mu}E_{\eta\varepsilon\rho}-g_{\eta\varepsilon}g^{\rho\mu}E_{\rho\varepsilon\lambda}=0$ (5) then $E_{\eta\varepsilon\rho}=0$.

Since the metric is a symmetric tensor, through summation over $\lambda = \mu$ in (5) we have $E_{\eta\delta\delta}=0$ and so (5) leads to $A_{\lambda\eta\rho}=A_{\eta\lambda\rho}$ and because $E_{\eta\varepsilon\rho}$ is antisymmetric in η,ρ with $A_{\eta\lambda\rho}=g_{\lambda\varepsilon}E_{\eta\varepsilon\rho}$ we obtain

$A_{\lambda\eta\rho}=-A_{\rho\eta\lambda}=-A_{\eta\rho\lambda}=A_{\lambda\rho\eta}$,

$A_{\lambda\eta\rho}=A_{\lambda\rho\eta}=A_{\rho\lambda\eta}=-A_{\eta\lambda\rho}=-A_{\lambda\eta\rho}=0$ and so $E_{\eta\varepsilon\rho}=0$.

Thus if Ω are antisymmetric in the Lorentz indices we have unique determined antisymmetric spin connection coefficients $\overline{\omega}_\mu^{ab}$ and $\widehat{\omega}_\mu^{ab}$ by the Euler-Lagrange equations $\frac{\delta S}{\delta\omega}=0$.

In the absence of fermions we have an unique torsion free antisymmetric spin connection and considering (3) we have

$$\epsilon^{abpq}\frac{\partial P_{EH}}{\partial\omega_\mu^{ab}}=\frac{1}{k_N^2}\epsilon^{\rho\nu\lambda\mu}(\omega_{\lambda a}^{p}e_\rho^{[q}e_\nu^{a]}-\omega_{\lambda a}^{q}e_\rho^{[p}e_\nu^{a]})$$

$$\epsilon^{abpq}d_\lambda\frac{\partial Q_{EH}}{\partial(\partial_\lambda\omega_\mu^{ab})}=\frac{1}{k_N^2}\epsilon^{\rho\nu\mu\lambda}(\partial_\lambda e_\rho^{[q}e_\nu^{p]}-e_\rho^{[q}\partial_\lambda e_\nu^{p]}) \text{ .}$$

The Euler-Lagrange equations

$\frac{\partial P_{EH}}{\partial\omega_\mu^{ab}}-d_\lambda\frac{\partial Q_{EH}}{\partial(\partial_\lambda\omega_\mu^{ab})}=0$ then lead to the equivalent relations

$$\epsilon^{\rho\lambda\nu\mu}(D_{[\lambda}e_{\nu]}^{p}e_\rho^{q}-D_{[\lambda}e_{\nu]}^{q}e_\rho^{p})=0$$

which are obviously satisfied by the unique torsion free antisymmetric spin connection that is

$$\overline{\omega}_\mu^{ab}=\frac{1}{2}e^{va}(\partial_\mu e_v^b-\partial_v e_\mu^b)+\frac{1}{2}e^{vb}(\partial_v e_\mu^a-\partial_\mu e_v^a)+\frac{1}{2}e^{\rho a}e^{\sigma b}(\partial_\sigma e_{\rho c}-\partial_\rho e_{\sigma c})e_\mu^c.$$

The infinitesimal Einstein transformation $x^\mu \to x'^\mu = x^\mu+\xi^\mu(x)$ gives on ω the transformation $\omega_\mu^{ab}\to\omega'^{ab}_\mu=\omega_\mu^{ab}+\delta_\xi\omega_\mu^{ab}$

$\delta_\xi\omega_\mu^{ab}=\xi^v\partial_v\omega_\mu^{ab}+\partial_\mu\xi^v\omega_v^{ab}$ and the infinitesimal local Lorentz transformation $x^a\to x'^a=x^a+\delta_{lL}x^a$ has

$\delta_{lL}x^a=\lambda^a{}_b(x)x^a$, $\Lambda=\exp(\frac{1}{2}\lambda^{ab}J_{ab})$, $S=S(\Lambda)=\exp(\frac{1}{4}\lambda^{ab}\sigma_{ab})$,

with $\sigma_{ab}=\frac{1}{2}[\gamma_a,\gamma_b]$, $\omega_\mu^{ab}\to\omega'^{ab}_\mu=\omega_\mu^{ab}+\delta_{lL}\omega_\mu^{ab}$ such that

$D'_\mu(\Lambda^a{}_b A^b)=\partial_\mu(\Lambda^a{}_b A^b)+\omega'^a_{\mu c}\Lambda^c{}_b A^b=\Lambda^a{}_b D_\mu A^b$ and so

$\delta_{lL}\omega_\mu^{ab}=-\partial_\mu\lambda^{ab}-\omega_{\mu c}^a\lambda^{cb}+\omega_{\mu c}^b\lambda^{ca}$

We can verify that we obtain the same compatible transformation by imposing the transformation for fermion fields

$D'_\mu(S\psi)=\partial_\mu(S\psi)+\frac{1}{4}\omega'^{ab}_\mu\sigma_{ab}(S\psi)=S D_\mu\psi$.

Following the steps above for the (4) equation with $E_{\rho\varepsilon\mu}=-E_{\mu\varepsilon\rho}$ we obtain

$2E_{\eta\delta\delta}=W_{\sigma\sigma\eta}$ (5)

$-g_{\varepsilon\lambda}E_{\eta\varepsilon\rho}+g_{\varepsilon\eta}E_{\lambda\varepsilon\rho}=g_{\rho\mu}W_{\lambda\mu\eta}-\frac{1}{2}g_{\rho\lambda}W_{\sigma\sigma\eta}+\frac{1}{2}g_{\rho\eta}W_{\sigma\sigma\lambda}=V_{\rho\lambda\eta}$.

With $A_{\lambda\eta\rho}=g_{\eta\varepsilon}E_{\lambda\varepsilon\rho}$ we have $A_{\lambda\eta\rho}=-A_{\rho\eta\lambda}$, $A_{\lambda\eta\rho}-A_{\eta\lambda\rho}=V_{\rho\lambda\eta}$,

$A_{\lambda\eta\rho}=-A_{\rho\eta\lambda}=-A_{\eta\rho\lambda}+V_{\lambda\eta\rho}=A_{\lambda\rho\eta}+V_{\lambda\eta\rho}=A_{\rho\lambda\eta}+V_{\eta\lambda\rho}+V_{\lambda\eta\rho}=$

$=-A_{\eta\lambda\rho}+V_{\eta\lambda\rho}+V_{\lambda\eta\rho}=-A_{\lambda\eta\rho}+V_{\rho\lambda\eta}+V_{\eta\lambda\rho}+V_{\lambda\eta\rho}$ and so

$A_{\lambda\eta\rho}=\frac{1}{2}(V_{\rho\lambda\eta}-V_{\lambda\rho\eta}+V_{\eta\lambda\rho})$

$E_{\lambda\varepsilon\rho}=g^{\varepsilon\eta}g_{\mu[\rho}W_{\lambda]\mu\eta}-\delta_{\varepsilon[\rho}W_{\lambda]\sigma\sigma}-\frac{1}{2}W_{\rho\varepsilon\lambda}$ (6) .

In a supergravity theory we have particles up to spin 2 (graviton has spin2) and therefore we must have a supersymmetry partner particle to the spin2 graviton which is a spin 3/2 particle, the gravitino. The gravitino can be

representd by a field $(\psi_\mu)_{\mu=\overline{0,d-1}}$ where μ is a covariance index and for any $\mu=\overline{0,d-1}$, ψ_μ are anticommuting Majorana spinors.
To have an irreducible spin 3/2 representation we impose the restriction $\gamma^\mu\psi_\mu=0$. Indeed, for $d=2n=4$ spacetime dimension, $(\psi_\mu)_\mu$ comes with $2^n d$ real degrees of freedom, that is $2^{n-1}d=8$ complex degrees of freedom from which the imposed restriction takes $2^n=4$ complex degrees of freedom, that is the gravitino has $2^{n-1}d-2^n=$ $2^{n-1}(d-2)=4$, the right number $2j+1=4$ complex degrees of freedom for a spin $j=3/2$ field.
We notice that the transition between covariance and Lorentz indices is $A_\mu=e^a_\mu A_a$ and so we defined $\gamma^\mu=e^\mu_a\gamma^a$ where γ^a are the Dirac gamma matrices.
The equation for a free massless on-shell Majorana spin 3/2 field $(\psi_\mu)_\mu$ satisfying $\overline{\psi}_\mu=\psi^T_\mu C$, $(C=-i\gamma^0\gamma^2$ for $d=4)$ and $\gamma^\mu\psi_\mu=0$ is the Rarita-Schwinger equation (in absence of gravitation $\gamma^\mu = \gamma^a$) $\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}\partial_\nu\psi_\mu=0$ which comes as Euler-Lagrange equation from the Rarita-Schwinger action
$S_{RS}=\frac{i}{2}\int d^4x\,\overline{\psi}_\mu\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}\partial_\nu\psi_\rho$ (the i coefficient is for the action to be real). We notice that the Rarita-Schwinger action for free Majorana spinors has a gauge freedom $\psi_\rho\to\psi_\rho+\partial_\rho\varphi$ so by gauge fixing this invariance we can have $\gamma^\rho\psi_\rho=0$ as requested for the irreducibility of the spin 3/2 representation.
The irreducibility restriction and the Rarita-Schwinger equations lead to $\sum_{\substack{\nu\neq\rho\\ \nu,\rho\neq\mu}}\gamma^\nu\gamma^\rho\partial_\nu\psi_\rho=0$ for any μ and with summation over μ we obtain $\sum_{\nu\neq\rho}\gamma^\nu\gamma^\rho\partial_\nu\psi_\rho=0$ and since $\gamma^\rho\psi_\rho=0$ we have $\sum_\nu\gamma^\nu\gamma^\nu\partial_\nu\psi_\nu=0$ that is $\partial^\nu\psi_\nu=0$. Also for any μ we have now

$$0=\sum_{\nu\neq\rho}\gamma^\mu\gamma^\nu\gamma^\rho\partial_\nu\psi_\rho-\sum_{\rho\neq\mu}\gamma^\mu\gamma^\mu\gamma^\rho\partial_\mu\psi_\rho-\sum_{\nu\neq\mu}\gamma^\mu\gamma^\nu\gamma^\mu\partial_\nu\psi_\mu=\gamma^\mu\gamma^\mu\gamma^\mu\partial_\mu\psi_\mu+$$
$$+\sum_{\nu\neq\mu}\gamma^\mu\gamma^\mu\gamma^\nu\partial_\nu\psi_\mu=\gamma^\nu\partial_\nu\psi^\mu=0\ .$$

Thus if the Rarita-Schwinger equation and the irreducibility restriction are satisfied then also we have $\gamma^\nu\partial_\nu\psi_\mu=0$ and $\partial^\nu\psi_\nu=0$.

In the general covariant with gravitation case we consider the action

$S=S_{EH}+S_{RS}$ where $S_{RS}=\frac{i}{2}\int d^4x(\det e)\overline{\psi}_\mu\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}D_\nu\psi_\rho$.

For $d=4$ we have $i(\det e)\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}=\epsilon^{\nu\mu\rho\sigma}\gamma_5\gamma_\sigma$ since

$e_a^{[\mu}e_b^\nu e_c^{\rho]}=\frac{1}{6}\epsilon^{\mu\nu\rho\sigma}e_\sigma^d\epsilon_{abcd}$ and $\gamma^{[a}\gamma^b\gamma^{c]}=\frac{i}{6}\gamma_5\gamma_d\epsilon^{abcd}$

Considering $\psi_\mu^*=i\gamma^0C^T\psi_\mu$, $\gamma^{\mu*}=-\gamma^0C\gamma^\mu C^{-1}\gamma^0$ we can verify that the supergravity action S is real and also $\overline{\Omega}_{\mu ab}$ and $\widehat{\Omega}_{\mu ab}$ are real and antisymmetric in a,b and so we have unique determined real and antisymmetric spin connection coefficients $\overline{\omega}$, $\widehat{\omega}$ (at least for $d=2,4,10$ for which $C=-C^T=-C^{-1}$).

For the supergravity action (in d = 4 case)

$$S=\frac{1}{8k_N^2}\int d^4x\,\epsilon_{abcd}\epsilon^{\rho\nu\mu\sigma}e_\rho^c e_\nu^d R_{\mu\sigma}^{ba}(\omega)+\frac{1}{2}\int d^4x\,\overline{\psi}_\mu\gamma_5\gamma_\sigma D_\nu\psi_\rho\epsilon^{\rho\nu\mu\sigma}$$

with $R_{\mu\sigma}^{ab}=\partial_\sigma\omega_\mu^{ab}-\partial_\mu\omega_\sigma^{ab}+\omega_{\sigma k}^a\omega_\mu^{kb}-\omega_{\mu k}^a\omega_\sigma^{kb}$ (7)

$$D_\nu\psi_\rho=\partial_\nu\psi_\rho+\frac{1}{4}\omega_\nu^{pq}\gamma^p\gamma^q\psi_\rho-\Gamma_{\nu\rho}^\lambda\psi_\lambda$$

we consider the variation under field variations given as

$$\delta\psi_\mu=\frac{\beta}{k_N}D_\mu\epsilon\ ,\ \delta e_\mu^a=i\frac{k_N\alpha}{2}\overline{\epsilon}\gamma^a\psi_\mu$$

$\delta\omega_\mu^{ab}=-\frac{\kappa}{4}\overline{\epsilon}\gamma_5\gamma_\mu\widetilde{\psi}^{ab}+\frac{\kappa}{8}\overline{\epsilon}\gamma_5(\gamma_c\widetilde{\psi}^{cb}e_\mu^a-\gamma_c\widetilde{\psi}^{ca}e_\mu^b)$ where β,α,κ are (8)
constants we will precise further and $\widetilde{\psi}^{ab}=\epsilon^{abcd}\psi_{cd}$,
$\psi_{cd}=e_c^\mu e_d^\nu(D_\mu\psi_\nu-D_\nu\psi_\mu)$ and $\epsilon=\epsilon(x)$ is a Majorana spinor local infinitesimal function variation parameter.

For A,B Majorana spinor functions we have $\overline{A}\gamma_5\gamma_\sigma B=\overline{B}\gamma_5\gamma_\sigma A$ and since $D_\lambda\left(\frac{\epsilon^{\rho\nu\mu\sigma}}{\det e}\right)=0$ and $\int d^4x(\det e)D_\mu F^\mu=0$ it follows

$$\int d^4x\,\epsilon^{\rho\nu\mu\sigma}\overline{D_\nu A_\rho}\gamma_5\gamma_\sigma B_\mu=-\int d^4x\Big(\partial_\nu e_\sigma^d\overline{A}_\rho\gamma_5\gamma_d B_\mu\epsilon^{\rho\nu\mu\sigma}+$$
$$+e_\sigma^d\overline{A}_\rho\gamma_5 D_\nu(\gamma_d B_\mu)\epsilon^{\rho\nu\mu\sigma}\Big)=-\int d^4x\Big(\partial_\nu e_\sigma^d\overline{A}_\rho\gamma_5\gamma_d B_\mu\epsilon^{\rho\nu\mu\sigma}-$$
$$-\frac{1}{4}\overline{A}_\rho\gamma_5[\gamma_d,\gamma_a\gamma_b]\omega_\nu^{ab}B_\mu e_\sigma^d\epsilon^{\rho\nu\mu\sigma}\Big)=$$

$$=-\int d^4x\left(D_\nu e^d_\sigma\right)\overline{A}_\rho\gamma_5\gamma_d B_\mu\epsilon^{\rho\nu\mu\sigma}-\int d^4x\overline{A}_\rho\gamma_5\gamma_\sigma D_\nu B_\mu\epsilon^{\rho\nu\mu\sigma}\qquad(9)\ .$$

We have also

$$D_\nu D_\rho\epsilon-D_\rho D_\nu\epsilon=R^{pq}_{\rho\nu}(\omega)\frac{1}{4}\gamma_p\gamma_q\epsilon$$

$$D_\nu D_\rho e^d_\sigma-D_\rho D_\nu e^d_\sigma=R^{dq}_{\rho\nu}(\omega)e_{\sigma q}$$

$$D_\nu D_\rho\gamma^c-D_\rho D_\nu\gamma^c=R^{cq}_{\rho\nu}(\omega)\gamma_q\ .$$

With the above relations, after some calculus we obtain

$$\frac{\delta S_{RS}}{\delta\psi}\delta\psi=-\frac{\beta}{2k_N}\int d^4x\epsilon^{\rho\nu\mu\sigma}\left(\overline{\psi}_\rho\gamma_5\gamma_d D_\mu\epsilon\right)\left(D_\nu e^d_\sigma\right)+$$

$$+\frac{\beta}{4k_N}\int d^4x\epsilon^{\rho\nu\mu\sigma}e_{\sigma d}\left(\overline{\psi}_\rho\gamma_5 R^{dq}_{\nu\mu}\gamma_q\epsilon\right)+$$

$$+\frac{\beta i}{8k_N}\int d^4x\epsilon_{dabc}\epsilon^{\rho\nu\mu\sigma}e^d_\sigma\left(\overline{\psi}_\rho\gamma^c\epsilon\right)R^{ba}_{\nu\mu}$$

$$\frac{\beta}{4k_N}\int d^4x\epsilon^{\rho\nu\mu\sigma}e_{\sigma d}\left(\overline{\psi}_\rho\gamma_5 R^{dq}_{\nu\mu}\gamma_q\epsilon\right)=$$

$$=-\frac{\beta}{2k_N}\int d^4x\epsilon^{\rho\nu\mu\sigma}D_\nu\left(\overline{\psi}_\rho\gamma_5\gamma_q\epsilon\right)D_\mu e^q_\sigma=$$

$$=-\frac{\beta}{2k_N}\int d^4x\epsilon^{\rho\nu\mu\sigma}\left(\left(\overline{D_\nu\psi_\rho}\gamma_5\gamma_q\epsilon\right)D_\mu e^q_\sigma+\left(\overline{\psi}_\rho\gamma_5\gamma_q D_\nu\epsilon\right)D_\mu e^q_\sigma\right)$$

$$\frac{\delta S_{EH}}{\delta e}\delta e=-\frac{i\alpha}{8k_N}\int d^4x\epsilon_{abcd}\epsilon^{\rho\nu\mu\sigma}e^d_\nu\left(\overline{\psi}_\rho\gamma^c\epsilon\right)R^{ba}_{\mu\sigma}$$

and so taking $\alpha=\beta$ we have

$$\frac{\delta S_{RS}}{\delta\psi}\delta\psi+\frac{\delta S_{EH}}{\delta e}\delta e=-\frac{\beta}{2k_N}\int d^4x\epsilon^{\rho\nu\mu\sigma}\left(\overline{\epsilon}\gamma_5\gamma_q D_\nu\psi_\rho\right)D_\mu e^q_\sigma\ .$$

Further we have

$$\frac{\delta S_{EH}}{\delta\omega}\delta\omega=-\frac{1}{4k_N^2}\int d^4x\epsilon_{abcd}\epsilon^{\rho\nu\mu\sigma}e^c_\rho e^d_\nu D_\sigma\left(\delta\omega^{ab}_\mu\right)=$$

$$=\frac{1}{2k_N^2}\int d^4x\epsilon_{abcd}\epsilon^{\rho\nu\mu\sigma}\delta\omega^{ab}_\mu e^d_\nu D_\sigma e^c_\rho$$

$$\epsilon_{abcd}\epsilon^{\rho\nu\mu\sigma}\delta\omega^{ab}_\mu=\kappa\overline{\epsilon}\gamma_5\gamma_c D_\nu\psi_\mu$$

$$\frac{\delta S_{RS}}{\delta\omega}\delta\omega=-\frac{i}{8}\int d^4x\epsilon^{\rho\nu\mu\sigma}\epsilon_{dpqr}e^d_\sigma\overline{\psi}_\mu\gamma^r\psi_\rho\delta\omega^{pq}_\nu=$$

$$=\frac{\kappa i}{8}\int d^4x\,\epsilon^{\rho\nu\mu\sigma}(\overline{\psi}_\mu\gamma^d\psi_\rho)(\overline{\epsilon}\gamma_5\gamma_d D_\sigma\psi_\nu)\ .$$

$$\frac{\delta S_{RS}}{\delta e}\delta e=\frac{i k_N\alpha}{4}\int d^4x\,\epsilon^{\rho\nu\mu\sigma}(\overline{\epsilon}\gamma^d\psi_\sigma)(\overline{\psi}_\mu\gamma_5\gamma_d D_\nu\psi_\rho)\ .$$

Let $(O_j)_j=(\boldsymbol{I},\gamma_5,(\gamma_d)_d,(\gamma_5\gamma_c)_c,(\gamma_{ab})_{a<b})$,

$(O^j)_j=(\boldsymbol{I},\gamma_5,(\gamma^d)_d,(\gamma^c\gamma_5)_c,(\gamma^b\gamma^a)_{a<b})$ complete set of 4x4 matrices

and for any $M=(M_{\alpha\beta})_{\alpha,\beta}\in M_{4\times4}(\mathbb{C})$ we have $M=\frac{1}{4}\mathrm{tr}(MO_j)O^j$

and so $\frac{1}{4}O_{j\delta\varepsilon}O^j_{\beta\alpha}=\delta_{\beta\varepsilon}\delta_{\alpha\delta}$ which for anticommuting spinors χ,ψ,φ

leads to $M\chi(\overline{\psi}N\varphi)=-\frac{1}{4}(MO_jN\varphi)(\overline{\psi}O^j\chi)$ for any $M,N\in M_{4\times4}(\mathbb{C})$.

Hence $\frac{\delta S_{RS}}{\delta e}\delta e=-\frac{i\alpha k_N}{4}\int d^4x\,\epsilon^{\rho\nu\mu\sigma}\frac{1}{4}(\overline{\epsilon}\gamma^d O_j\gamma_5\gamma_d D_\nu\psi_\rho)(\overline{\psi}_\mu O^j\psi_\sigma)=$

$=\frac{i k_N\alpha}{8}\int d^4x\,\epsilon^{\rho\nu\mu\sigma}(\overline{\epsilon}\gamma_5\gamma_d D_\nu\psi_\rho)(\overline{\psi}_\mu\gamma^d\psi_\sigma)$ and so taking

$\alpha k_N=\beta k_N=-\kappa$ we will have

$$\delta S=\frac{\delta S_{RS}}{\delta e}\delta e+\frac{\delta S_{RS}}{\delta\omega}\delta\omega+\frac{\delta S_{EH}}{\delta\omega}\delta\omega+\frac{\delta S_{RS}}{\delta\psi}\delta\psi+\frac{\delta S_{EH}}{\delta e}\delta e=0$$

which proves the invariance of the supergravity (7) action in the first order formalism under the supersymmetry variations (8) with $\alpha=\beta=1$, $\kappa=-k_N$.
In the second order formalism we still have

$\delta S=\frac{\delta S}{\delta\psi}\delta\psi+\frac{\delta S}{\delta e}\delta e+\frac{\delta S}{\delta\omega}\delta\omega=0$ with field variations defined by (8)

but since $\frac{\delta S}{\delta\omega}=0$ we will have

$\delta\hat{S}=\frac{\delta\hat{S}}{\delta\psi}\delta\psi+\frac{\delta\hat{S}}{\delta e}\delta e=\frac{\delta S}{\delta\psi}\delta\psi+\frac{\delta S}{\delta e}\delta e=0$ with

$$\delta\psi_\mu=\frac{1}{k_N}D_\mu\epsilon\quad,\quad\delta e^a_\mu=i\frac{k_N}{2}\overline{\epsilon}\gamma^a\psi_\mu\qquad(10)$$

proving the invariance of the supergravity action (7) in the second order formalism under supersymmetry variations (10).
The gravitino field in the above irreducible Majorana spin 3/2 representation is at the same time the supersymmetry fermionic partner of the graviton field represented by the vielbein and the gauge field of the localized supersymmetry transformation.

The supersymmetry generators are $(Q_\alpha)_\alpha$ which are in a Majorana spinor representation of the Lorentz group and the supersymmetry correspondence is $Q_\alpha(e_\nu^a)=C_{\alpha\beta}\gamma^a_{\beta\delta}\psi^\delta_\nu$, $Q_\alpha((\psi^\delta_\nu)_{\delta,\nu})=\bar{0}$ with $(e^a_\nu)_{a,\nu}$ the vielbein field and $(\psi^\delta_\nu)_\delta$ Majorana spinor fields satysfying $\gamma^\nu\psi_\nu=0$ so that $\delta_\epsilon e^a_\nu=\epsilon^\alpha Q_\alpha(e^a_\nu)$ and considering that Q_α act on Lorentz vector and Dirac spinor indexes *a* and *δ* which are for the fundamental vector and respective Majorana spinor representations on which we have no action of the translation generator of the Poincare group we can verify that in that representation the supersymmetry graded Lie algebra commutation and anti-commutation relations are satisfied.
The on-shell gravitino field satisfies the equivalent motion equations

$$-\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\omega^{ab}_\nu C\gamma^c\psi_\rho+\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}e^d_\sigma C\gamma_5\gamma_d\partial_\nu\psi_\rho+$$

$$+\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}C\gamma_5\gamma_d\partial_\nu(e^d_\sigma\psi_\rho)+iC\gamma^\mu\lambda=0 \qquad (11)$$

and the restriction $\gamma^\mu\psi_\mu=0$ (12)
where $\lambda=\lambda(x)$ is a Majorana spinor field Lagrange multiplier parameter. The gravitino field starts with $2^n d$, ($d = 2n$ space-time dimension) real degrees of freedom and the relations (11), (12) relate one half of the spinor components to the other half and represent therefore $2^{n-1}(d+1)$ independent equations. The Lagrange multiplier field has 2^n real degrees of freedom. Thus the on-shell gravitino has $2^n d-2^{n-1}(d+1)-2^n=2^{n-1}(d-3)=2$ (for $d=4$) real degrees of freedom.
In the weak gravitational field approximation we have a coordinate system $(x^\mu)_\mu$ choice such that $g_{\mu\nu}=\eta_{\mu\nu}+k_N h_{\mu\nu}+O(\varepsilon^2)$ with $h_{\mu\nu}\in O(\varepsilon)$, $h_{\mu\nu}(\bar{0})=0$, $h_{\mu\nu}=h_{\nu\mu}$.
The action (in absence of fermions) is the Hilbert-Einstein action

$$S=\frac{1}{2k_N^2}\int d^4x\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}$$

with $(R_{\mu\nu})_{\mu,\nu}$ the Ricci tensor and $g=\det(g_{\mu\nu})_{\mu,\nu}$ and is invariant at first order under Einstein infinitesimal transforms $x^\mu\rightarrow x'^\mu=x^\mu+\xi^\mu(x)$ and we have $g^{\mu\nu}=\eta^{\mu\nu}-k_N h^{\mu\nu}+O(\varepsilon^2)$, $h_{\mu\nu}\rightarrow h'_{\mu\nu}=h_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu$, $\sqrt{-g}=1+(1/2)k_N h^\mu_\mu+O(\varepsilon^2)$

with $h^{\mu\nu}=\eta^{\mu\lambda}\eta^{\nu\sigma}h_{\lambda\sigma}$ and real constants a,b,c,d such that

$$S=\int d^4x\left(a\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu}+b\partial_\lambda h_\mu^\mu\partial^\lambda h_\nu^\nu+c\partial_\lambda h^{\lambda\nu}\partial^\mu h_{\mu\nu}+d\partial^\mu h_\lambda^\lambda\partial^\nu h_{\mu\nu}\right) \quad (13)$$

at first order in ε.

We can determine *a, b, c, d* from the condition that *S* is invariant under an arbitrary Einstein infinitesimal transformation.

For S as in (13) and $h=h_\lambda^\lambda=\eta^{\lambda\mu}h_{\mu\lambda}$ integrating by parts we obtain under $\delta x^\mu=\xi^\mu(x)$ that

$$\delta S=\int d^4x\,\xi^\nu\left((4a+2c)\partial^2\partial^\mu h_{\mu\nu}+(4b+2d)\partial^2\partial_\nu h+(2c+2d)\partial_\nu\partial^\mu\partial^\lambda h_{\mu\lambda}\right)$$

Taking $\hat{F}(p)=\int\exp(ip_\mu x^\mu)F(x)d^4x$, $\xi^\nu(x)=\Re\left(\epsilon^\nu\exp(ip_\mu x^\mu)\right)$, where $(\epsilon^\nu)_\nu\in\mathbb{C}^4$ and $(p_\nu)_\nu\in\mathbb{R}^4$ are arbitrary constants, the invariance of S under Einstein infinitesimal transformations imposes

$$\Re\left((4a+2c)\epsilon^\nu p^2p^\mu\hat{h}_{\mu\nu}+(4b+2d)\epsilon^\nu p^2p_\nu\hat{h}+(2c+2d)\epsilon^\nu p_\nu p^\mu p^\lambda\hat{h}_{\mu\lambda}\right)=0$$

with arbitrary $(\epsilon^\nu,p_\nu,h_{\mu\nu})_{\mu,\nu}$ and so taking first $\epsilon^\nu p_\nu=0$ it follows $4a+2c=0$ and taking now $p^2=0$ it follows $2c+2d=0$ and finallly $4b+2d=0$. Therefore we must have

$$S=K\int d^4x\left(\frac{1}{2}\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu}-\frac{1}{2}\partial_\lambda h\partial^\lambda h-\partial_\lambda h^{\lambda\nu}\partial^\mu h_{\mu\nu}+\partial^\mu h\partial^\nu h_{\mu\nu}\right)$$

Turning to a general gauge transformation $\delta x^\mu=\xi^\mu(x)$ defining $\bar{h}_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h$ we have $\bar{h}_{\mu\nu}\rightarrow\bar{h}'_{\mu\nu}=\bar{h}_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu-\eta_{\mu\nu}\partial^\sigma\xi_\sigma$ so at first order we have $\partial^\nu\bar{h}_{\mu\nu}\rightarrow\partial^\nu\bar{h}_{\mu\nu}+\Box\xi_\mu$.

Taking ξ such that $\Box\xi_\mu=-\partial^\nu\bar{h}_{\mu\nu}$ we can have in the so called Donder gauge that $\partial^\nu h_{\mu\nu}=\frac{1}{2}\partial_\mu h$ and the action in the Donder gauge becomes $S=K\int d^4x\left(\frac{1}{2}\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu}-\frac{1}{2}\partial_\lambda h\partial^\lambda h-\partial_\lambda h^{\lambda\nu}\partial^\mu h_{\mu\nu}+\right.$

$$\left.+\partial^\mu h\partial^\nu h_{\mu\nu}+\left(\partial^\mu h_{\mu\nu}-\frac{1}{2}\partial_\nu h\right)^2\right)=\frac{1}{2}K\int d^4x\left(\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu}-\frac{1}{2}\partial_\lambda h\partial^\lambda h\right) .$$

Hence in the weak gravitational field we have an action $S_{tot}=S+S_{mat}$, with $S_{mat}=\int d^4x\,L_{mat}$ so the motion equations to first order in the Donder gauge are $K\left(-\Box h_{\mu\nu}+\frac{1}{2}\eta_{\mu\nu}\Box h\right)=\frac{1}{2}T_{\mu\nu}$ where

$T_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta L_{mat}}{\delta g_{\mu\nu}}$ so that $\delta S_{mat}=\int d^4 x\frac{\delta L_{mat}}{\delta g_{\mu\nu}}\delta g_{\mu\nu}$.

Computing the Ricci tensor $R_{\mu\nu}=-R^{\rho}_{\mu\rho\nu}$
and the Ricci scalar $R=g^{\mu\nu}R_{\mu\nu}$ in the weak field approximation we have

$R_{\mu\nu}=\frac{1}{2}\left(-\Box h_{\mu\nu}+\partial_\mu\partial_\lambda h^\lambda_\nu+\partial_\nu\partial_\lambda h^\lambda_\mu-\partial_\mu\partial_\nu h^\lambda_\lambda\right)+O\left(\varepsilon^2\right)$

and so in the Donder gauge we have

$$R_{\mu\nu}=-\frac{1}{2}\Box h_{\mu\nu}+O\left(\varepsilon^2\right)\ ,\ R=-\frac{1}{2}\Box h+O\left(\varepsilon^2\right)$$

The motion equations must match the Einstein equations
$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=k_N^2 T_{\mu\nu}$ and so we conclude that we must have
and the action in the Donder gauge in absence of matter is
$S=\frac{1}{8k_N^2}\int d^4 x\left(\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu}-\frac{1}{2}\partial_\lambda h\partial^\lambda h\right)$ with the motion equation
for the on-shell graviton in the Donder gauge being a wave equation:
$\Box\bar{h}_{\mu\nu}=0$ where $\bar{h}_{\mu\nu}=h_{\mu\nu}-\eta_{\mu\nu}\frac{1}{2}h$, $\partial^\nu\bar{h}_{\mu\nu}=0$

We have $\bar{h}^\mu_\mu=h^\mu_\mu-2h=-h$ and so on-shell in the Donder gauge $\Box h=0$
having an expansion $h=\Re\sum_k b_k\exp\left(i k_\lambda x^\lambda\right)$ with $k=\left(k_\lambda\right)_\lambda\in\mathbb{R}^4$,
$b_k\in\mathbb{C}$, $k^2=k^\lambda k_\lambda=0$ such that $b_{\bar{0}}=0$ (since $h\left(\bar{0}\right)=0$),
$\left(k_j\right)_{j=\overline{1,3}}\in\frac{2\pi}{L}\mathbb{Z}^3$ considering the graviton field confined in a cubic
box of edgelenght L centered at $x=0$.

Therefore taking in a Donder gauge the additional Einstein transformation
$\delta x^\mu=\xi^\mu=\Re\sum_k a^\mu_k\exp\left(i k_\lambda x^\lambda\right)$ with $a^\mu_{\bar{0}}=0$, $2 i k_\mu a^\mu_k+b_k=0$ because
$\Box\xi^\mu=0$ we are still in a Donder gauge and in this gauge we have also
that $h=0$, $\partial^\nu h_{\mu\nu}=0$ and the motion equation is $\Box h_{\mu\nu}=0$ leading to
a expansion $h^{\lambda\nu}=\sum_k a^{\lambda\nu}_k\exp\left(i k_\lambda x^\lambda\right)$ with $k_\lambda a^{\lambda\nu}_k=0$, $a^{\lambda\nu}_k=a^{\nu\lambda}_k$.

Because $k^2=0$ for each k we can have frames that conserve
the $a^{\lambda\nu}_k$ coefficients in which $\left(k_\lambda\right)_\lambda=\left(1,\pm1,0,0\right)$ and so λ,ν in
$a^{\lambda\nu}_k$ run over $d-2=2$ transverse to k directions and that, together

with the tracelessness condition $a_k^{\mu\nu} g_{\mu\nu}=0$, $(h=0)$ reduces the the degrees of freedom of the on-shell graviton to $\frac{(d-1)(d-2)}{2}-1=2.$

A gravitational wave has two polarization directions, transverse to the propagation direction.

The of-shell graviton field has, by gauge fixing, $k_\lambda a_k^{\lambda\nu}=0$(with not necessarily $k^2=0$) and $g_{\mu\nu} a_k^{\mu\nu}=0$, having $\frac{d(d+1)}{2}-d-1=5=2j+1$, the right number of degrees of freedom in a spin $j=2$ irreducible representation.

We notice that for *d* = 4 , the *N* = 1 supergravity on-shell with only one gravitino field of spin 3/2 and one graviton field of spin 2 represented as the vielbein, the fermionic degrees of freedom (= 2, on-shell gravitino) are equal to the bosonic degrees of freedom (=2 , on-shell graviton) as it should be, considering the correspondence between bosons and fermions in a supersymmetric theory.

A *N*=2 gravitino fields gauged supergravity theory has two gravitino fields $(\psi_\mu^j)_\mu$, $j=1{,}2$ with ψ_μ^j Majorana spinors satisfying the irreducibility restriction $\gamma^\mu \psi_\mu^j=0$, the graviton field as a vielbein $(e_\mu^a)_{a,\mu}$ and a real covariant gauge field $(A_\mu)_\mu$ with field strenght $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, a localized symmetry parametrized by real infinitesimal $\theta=\theta(x)$ defined as $\delta\psi_\mu^1=-\theta\psi_\mu^2$, $\delta\psi_\mu^2=\theta\psi_\mu^1$ corresponding to a $\psi_\mu \rightarrow \exp(i\theta)\psi_\mu$ with $\psi_\mu=\psi_\mu^1+i\psi_\mu^2$ symmetry and the gauge field variation $\delta A_\mu=-\frac{1}{g}\partial_\mu\theta$ (g a real constant), defining the covariant derivatives for the gravitino fields $\widetilde{D}_\mu\psi_\nu=D_\mu\psi_\nu+i g A_\mu\psi_\nu$.

We notice that since the joined components of ψ_μ^1,ψ_ν^2 are all anticommuting, the joined components of ψ_μ , $\overline{\psi}_\nu$ are all anticommuting.

For $d=4$ we will have the gauged supergravity action $S=S_{EH}+S_{RS}+S_A$ with

$$S_{EH}=\frac{1}{8k_N^2}\int d^4x\,\epsilon_{abcd}\epsilon^{\rho\nu\mu\sigma} e_\rho^c e_\nu^d R_{\mu\sigma}^{ba}(\omega)$$

$$S_{RS}=\frac{1}{2}\int d^4x\,\overline{\psi}^k_\mu\gamma_5\gamma_\sigma\widetilde{D}_\nu\psi^k_\rho\epsilon^{\rho\nu\mu\sigma}=\frac{1}{2}\Re\int d^4x\,\overline{\psi}_\mu\gamma_5\gamma_\sigma\widetilde{D}_\nu\psi_\rho\ ,$$

$$S_A=-\frac{1}{4}\int d^4x(\det e)F_{\mu\nu}F^{\mu\nu}.$$

(To make the action invariant under general coordinate transforms, even under orientation changing we multiply the above Lagrangian density with $\mathrm{sign}(\det e)$) .

In flat space-time, the motion equations for the gauge field are $\Box A_\mu=0$ in the $\partial^\nu A_\nu=0$ gauge , and so the gauge field has on-shell $d-2=2$ polarization directions which are transverse to propagation direction.

The gauged supergravity has supersymmetry given by supersymmetry generators $(Q^j_\alpha)_\alpha$, $j=1,2$ that are in a Majorana spinor representation, $Q^j_\alpha(e^a_\nu)=i\frac{k_N}{2}C_{\alpha\beta}\gamma^a_{\beta\delta}\psi^{j\delta}_\nu$ so that the supersymmetry invariance transformation has two Majorane spinor parameters $\epsilon^j=(\epsilon^{j\delta})_\delta$ or equivalent one spinor parameter $\epsilon=\epsilon^1+i\epsilon^2=\epsilon(x)$ and is defined as

$$\delta_\epsilon e^a_\nu=\epsilon^{k\alpha}Q^k_\alpha(e^a_\nu)=i\frac{k_N}{2}\overline{\epsilon}^k\gamma^a\psi^k_\nu=\Re\left(i\frac{k_N}{2}\overline{\epsilon}\gamma^a\psi_\nu\right)\qquad(14)$$

$$\delta_\epsilon\psi_\nu=\frac{1}{k_N}\widetilde{D}_\nu\epsilon\qquad(15)\ .$$

We have on-shell four bosonic degrees of freedom (two from the on-shell graviton and two from the on-shell gauge field) and four fermionic degrees of freedom (the two on-shell gravitinos) which is consistent with the fermion-boson supersymmetry correspondence.

Because the the invariance of the ungauged theory is already proven, to prove the invariance of the gauged supergravity action under the (14), (15) variations of the fields, we have to show only that

$$\delta S=\frac{1}{2k_N}\Re\int d^4x\,\epsilon^{\rho\nu\mu\sigma}g\big(-iA_\mu\overline{\epsilon}\gamma_5\gamma_\sigma D_\nu\psi_\rho+i\overline{\psi}_\mu\gamma_5\gamma_\sigma D_\nu(A_\rho\epsilon)+ \qquad(16)$$
$$+i\overline{D_\mu\epsilon}\gamma_5\gamma_\sigma A_\nu\psi_\rho+i\overline{\psi}_\mu\gamma_5\gamma_\sigma A_\nu D_\rho\epsilon\big)=0\ .$$

For φ,χ spinors with anticommuting $\varphi,\chi,\overline{\varphi},\overline{\chi}$ components we have $(i\overline{\varphi}\gamma_5\gamma_\sigma\chi)^*=-i\overline{\chi}\gamma_5\gamma_\sigma\varphi$, $(\overline{\varphi}\gamma^d\chi)^*=\overline{\chi}\gamma^d\varphi$, $(\overline{\varphi}\chi)^*=-\overline{\chi}\varphi$ and so considering (9) we obtain

$\delta S = E + B$ with

$$E = \frac{1}{k_N} \Re \int d^4 x\, i\, g (\overline{\psi}_\mu \gamma_5 \gamma_\sigma \partial_\nu A_\rho \epsilon) \epsilon^{\rho\nu\mu\sigma} \ ,$$

$$B = -\frac{1}{2 k_N} \Re \int d^4 x\, i\, g (D_\nu e_\sigma^d)(\overline{\psi}_\rho \gamma_5 \gamma_d A_\mu \epsilon) \epsilon^{\rho\nu\mu\sigma} \ .$$

We have $D_\nu e_\sigma^d = D_\nu(\overline{\omega}) e_\sigma^d + \hat{\omega}_\nu^{dq} e_{\sigma q} = \hat{\omega}_\nu^{dq} e_{\sigma q}$ and so after some calculus

$$B = -\frac{1}{2 k_N} \Re \int d^4 x\, i\, g (\overline{\psi}_\rho \gamma_5 \gamma_d A_\mu \epsilon) e^{\lambda d} \frac{1}{2} (g_{\nu\lambda} W_{\sigma\alpha\alpha} - g_{\alpha\lambda} W_{\sigma\alpha\nu}) \epsilon^{\rho\nu\mu\sigma} \ ,$$

where $W_{\lambda\mu\eta} = e_\lambda^a e_\eta^b \hat{\Omega}_{\mu a b}$

$$\hat{\Omega}_{\mu a b} = \frac{2 k_N^2}{\det e} \Re \left(\frac{-\partial L_F}{\partial \omega_\mu^{ab}} \right) = -\frac{2 k_N^2}{\det e} \frac{1}{16} \Re (\overline{\psi}_\nu \gamma_5 \gamma_d [\gamma_a, \gamma_b] \psi_\rho) =$$

$$= -\frac{i k_N^2}{8 \det e} \epsilon_{dabc} \epsilon^{\rho\nu\mu\sigma} e_\sigma^d (\overline{\psi}_\nu \gamma^c \psi_\rho) \ .$$

Therefore

$$\epsilon^{\rho\nu\mu\sigma} e^{\lambda d} g_{\nu\lambda} W_{\sigma\alpha\alpha} = -\frac{i k_N^2}{8 \det e} \epsilon^{\rho\nu\mu\sigma} e_\nu^d e_\sigma^a e_\alpha^b \epsilon_{eabc} (\overline{\psi}_\delta \gamma^c \psi_\beta) e_\kappa^e \epsilon^{\beta\delta\alpha\kappa} =$$

$$= -i \frac{k_N^2}{8} \epsilon^{\rho\nu\mu\sigma} e_\nu^d \epsilon_{\sigma\varepsilon\alpha\kappa} \epsilon^{\beta\delta\alpha\kappa} e_c^\varepsilon (\overline{\psi}_\delta \gamma^c \psi_\beta) = 0$$ because $\gamma^\delta \psi_\delta = \gamma^\beta \psi_\beta = 0$.

Considering again that $\gamma^\mu \psi_\mu = 0$ we derive

$$\Re (i\, g (\epsilon^{\rho\nu\mu\sigma} e^{\lambda d} g_{\alpha\lambda} W_{\sigma\alpha\nu} (\overline{\psi}_\rho \gamma_5 \gamma_d A_\mu \epsilon))) =$$

$$\Re (\epsilon^{\rho\nu\mu\sigma} i\, g (\overline{\psi}_\rho \gamma_5 \gamma_d A_\mu \epsilon) e_\alpha^d e_\sigma^a e_\nu^b \left(-\frac{i k_N^2}{8 \det e} \right) (\overline{\psi}_\delta \gamma^c \psi_\beta) e_\kappa^e \epsilon_{eabc} \epsilon^{\beta\delta\alpha\kappa}) =$$

$$= -\frac{k_N^2}{4} \Re (g (\overline{\psi}_\nu \gamma^e \psi_\sigma)(\overline{\psi}_\rho \gamma_5 \gamma_e A_\mu \epsilon))$$

Under a gauge transformation

$\psi_\mu \rightarrow \exp(i g \theta) \psi_\mu$, $\epsilon \rightarrow \exp(i g \theta) \epsilon$, $A_\mu \rightarrow A_\mu - \partial_\mu \theta$ for the fields and θ decreasing sufficiently fast to zero at infinity, $\delta S, E,$ and subsequently B are invariant: $\delta S \rightarrow \delta S_\theta = \delta S$, $E \rightarrow E_\theta = E$, $B \rightarrow B_\theta = B$ and therefore $B = B_\theta = \int d^4 x A_\mu H^\mu - \int d^4 x\, \theta\, \partial_\mu H^\mu$ with

$$H^\mu = \Re \left(\frac{-g k_N}{8} (\overline{\psi}_\nu \gamma^e \psi_\sigma)(\overline{\psi}_\rho \gamma_5 \gamma_e \epsilon) \right)$$

For sufficiently fast decreasing to zero fields $A_\mu, \psi\, su\, v, \partial_\lambda \psi_v, \epsilon, \partial_\lambda \epsilon$ we define sufficiently fast decreasing to zero function θ as $\theta = K\partial_\mu H^\mu$ with $K = \left(\int d^4 x\, A_\mu H^\mu\right)\left(\int d^4 x (\partial_\mu H^\mu)^2\right)^{-1}$. (for the most ψ , ϵ we can assume $\partial_\mu H^\mu$ not identical zero) and we obtain $B = B_\theta = 0$.

Hence the remaining expression, considering the above derivations is $\delta S = A = \frac{1}{k_N}\Re\int d^4 x\, i g(\overline{\psi}_\mu \gamma_5 \gamma_\sigma \epsilon)\epsilon^{\rho\nu\mu\sigma}\partial_\nu A_\rho = \int d^4 x\, A_\rho F^\rho$ where $F^\rho = \frac{1}{k_N}\Re(i g \partial_\nu(\overline{\psi}_\mu \gamma_5 \gamma_\sigma \epsilon))\epsilon^{\rho\nu\mu\sigma}$.

We assume further that F^ρ is analytical and A_ρ, F^ρ square integrable integrable functions on $\mathbb{R}^4$. (and obviously for the above relation to work we will suppose that $A_\rho(\overline{\psi}_\mu \gamma_5 \gamma_\sigma \epsilon)\epsilon^{\rho\nu\mu\sigma}$ decreases faster than $\|x\|^{-3/2}$ at infinity). Under a gauge transformation $\psi_\mu \rightarrow \exp(i\theta g)\psi_\mu$, $\epsilon \rightarrow \exp(i\theta g)\epsilon$, $A_\mu \rightarrow A_\mu - \partial_\mu \theta$ we have $\delta S \rightarrow \delta S_\theta$ and by gauge invariance $\delta S = \delta S_\theta$.

We notice that the variation δS is invariant under a gauge transformation with a general $\theta \in C^1(\mathbb{R}^4)$ only in its initial form (16), where we have no problems with the behaviour of θ at infinity. The the only concern when deriving the equivalent forms of δS is when we apply integration by parts, which involves behaviour of $\partial\theta$ at infinity. Doing that with with regard at the infinity for $\partial\theta$ we will have

$$\delta S = \delta S_\theta = \lim_{R\rightarrow\infty}\frac{1}{2k_N}\Re\left(i g\left(\int_{\|x\|=R}\partial_\rho\theta(\overline{\psi}_\mu \gamma_5 \gamma_\sigma \epsilon)\epsilon^{\rho\nu\mu\sigma} n_\nu\, d\sigma(x) + \right.\right.$$

$$-2\left(\int_{\|x\|=R}\partial_\rho\theta(\overline{\psi}_\mu \gamma_5 \gamma_\sigma \epsilon)\epsilon^{\rho\nu\mu\sigma} n_\nu\, d\sigma(x) + \int_{\|x\|<R} A_\rho \partial_\nu(\overline{\psi}_\mu \gamma_5 \gamma_\sigma \epsilon)\epsilon^{\rho\nu\mu\sigma} d^4 x\right) +$$

$$\left.\left. +2\int_{\|x\|=R}\theta\partial_\nu(\overline{\psi}_\mu \gamma_5 \gamma_\sigma \epsilon)\epsilon^{\rho\nu\mu\sigma} n_\rho\, d\sigma(x)\right)\right)$$

Also by flux-divergence theorem we have

$$\int_{\|x\|=R}\theta\partial_\nu(\overline{\psi}_\mu \gamma_5 \gamma_\sigma \epsilon)\epsilon^{\rho\nu\mu\sigma} d\sigma(x) - \int_{\|x\|=R}\partial_\rho\theta(\overline{\psi}_\mu \gamma_5 \gamma_\sigma)\epsilon^{\rho\nu\mu\sigma} n_\nu\, d\sigma(x) =$$

$$= \int_{\|x\|=R}\partial_\nu(\theta(\overline{\psi}_\mu \gamma_5 \gamma_\sigma \epsilon)\epsilon^{\rho\nu\mu\sigma} n_\rho\, d\sigma(x)) = 0$$ and so

$$\delta S=\delta S_\theta=\lim_{R\to\infty}\frac{1}{2k_N}\Re\left(ig\left(\int_{\|x\|=R}\theta\,\partial_\nu(\overline{\psi}_\mu\gamma_5\gamma_\sigma\epsilon)\epsilon^{\rho\nu\mu\sigma}n_\rho\,d\sigma(x)-\right.\right.$$
$$\left.\left.-\int_{\|x\|<R}2A_\rho\partial_\nu(\overline{\psi}_\mu\gamma_5\gamma_\sigma\epsilon)\epsilon^{\rho\nu\mu\sigma}d^4x\right)\right)$$

So we look for θ satisfying

$\int_{\|x\|=R}\theta F^\rho n_\rho\,d\sigma(x)=2\int_{\|x\|<R}d^4x\,A_\rho F^\rho$ for arbitrary R . ($n_\rho(x)=\frac{x^\rho}{\|x\|}$).

where $F^\rho=\partial_\nu(\overline{\psi}_\rho\gamma_5\gamma_\sigma\epsilon)$ is analytical and square integrable on $\mathbb{R}^4$

Consider the function

$$C(R)=\int_{\|x\|=R}(F^\rho n_\rho)^2 d\sigma(x)=R^3\int_{\|x\|=1}(F^\rho(Rx)n_\rho)^2 d\sigma(x)\ .$$

Since F^ρ is analytical we can find $r_n,\varepsilon_n\in\mathbb{R}_+$ with $r_{n+1}>r_n$

$\lim_{n\to\infty}r_n=\infty$, $n\in\mathbb{N}$, $K_n=\{x\in\mathbb{R}^4 \mid \|x\|\in(r_n-\varepsilon_n,r_n+\varepsilon_n)\}$,

$K_n\cap K_m=\emptyset$ for $m\neq n$, $K=\bigcup_n K_n$

such that $C(R)\neq 0$ for $R\in\mathbb{R}\setminus\bigcup_n(r_n-\varepsilon_n,r_n+\varepsilon_n)$ and

$\int_K|A_\rho F^\rho|d^4x<\varepsilon$.

We can now define $\theta:\mathbb{R}^4\setminus K\to\mathbb{R}$, $\theta(Rx)=\lambda(R)F^\rho(Rx)n_\rho(x)$ for

$\|x\|=1$, $Rx\in\mathbb{R}^4\setminus K$, $\lambda(R)=\frac{2}{C(R)}\int_{\|x\|<R}A_\rho F^\rho d^4x$

and we have $\theta\in C^\infty(\mathbb{R}^4\setminus K)$, $\int_{\|x\|=R}\theta F^\rho n_\rho\,d\sigma(x)=2\int_{\|x\|<R}A_\rho F^\rho d^4x$

We can now also take

$\theta_n\in C^1(\overline{K_n})$ with $\theta(x)=\theta_n(x)$ for $x\in\partial K_n$ and

$\frac{d}{dR}\theta_n(Rx)=\frac{d}{dR}\theta(Rx)$ for $\|x\|=1$ and $R\in\{r_n-\varepsilon_n,r_n+\varepsilon_n\}$.

Then for $\hat{\theta}(y)=\begin{cases}\theta(y) & \text{for } y\in\mathbb{R}^4\setminus K\\ \theta_n(y) & \text{for } y\in K_n\ ,\ n\in\mathbb{N}\end{cases}$ we have $\hat{\theta}\in C^1(\mathbb{R}^4)$

$$\delta S=\delta S_{\hat{\theta}}=\lim_{R\to\infty}\frac{1}{2}\left(\int_{\|x\|=R}\hat{\theta}F^\rho n_\rho\,d\sigma(x)-\int_{\|x\|<R}2A_\rho F^\rho d^4x\right)=0\ .$$

Thus under some assumptions about the fields F^ρ being analytical function and A_ρ , F^ρ well behave at infinity (for example decreasing faster then the inverse of the squared norm in four dimensions) the gauged supergravity action is invariant under the variations (14), (15) .

Consider now an additional field variation in the gauged supergravity theory given as $\delta\psi_\mu = i g \gamma_\mu \epsilon$ and then we have the corresponding variation of the action

$$\delta S = \frac{1}{2k_N}\Re\int d^4x\, i g\left(\overline{\psi}_\mu \gamma_5 \gamma_\sigma \widetilde{D}_\nu(\gamma_\rho \epsilon) - \overline{\epsilon}\gamma_5\gamma_\sigma\gamma_\rho \widetilde{D}_\nu \psi_\mu\right)\epsilon^{\rho\nu\mu\sigma}$$

With $\omega = \overline{\omega} + \widehat{\omega}$ we have that $\widehat{\omega}$ is quadratic in ψ and so the non-linear in ψ terms of δS are (after some calculus) given by

$$A = -\frac{1}{16k_N}\int d^4x\, \Re\left(g\, \overline{i\gamma_\rho\epsilon}\,\gamma_5\{\gamma_d,[\gamma_a,\gamma_b]\}\psi_\mu\right)\widehat{\omega}_\nu^{ab} e_\sigma^d \epsilon^{\rho\nu\mu\sigma} =$$

$$= \frac{1}{4k_N}\int d^4x\, B \text{ with } B = \Re\left(i g\, \overline{\delta\psi}_\rho \gamma^c \psi_\mu\right)\epsilon_{dabc} e_\sigma^d \widehat{\omega}_\nu^{ab} \epsilon^{\rho\nu\mu\sigma}$$

Because $\widehat{\omega}_\lambda^{ab} e_a^\lambda = e_a^\lambda g_{\lambda\varepsilon} e^{\mu a} e^{\rho b} E_{\mu\varepsilon\rho} = \frac{1}{2} e^{\rho b} W_{\rho\varepsilon\varepsilon} =$

$$= -\frac{i k_N^2}{16\det e}\epsilon_{kdcl}(\overline{\psi}_\alpha \gamma^l \psi_\beta)\epsilon^{\beta\alpha\varepsilon\gamma} e_\varepsilon^c e_\gamma^k \eta^{bd} =$$

$$= -\frac{i k_N^2}{16}\epsilon_{\varepsilon\gamma\nu\rho} e_d^\nu e_l^\rho \epsilon^{\varepsilon\gamma\beta\alpha}\overline{\psi}_\alpha\gamma^l\psi_\beta = -\frac{i k_N^2}{4} e_d^{[\beta} e_l^{\alpha]}\overline{\psi}_\alpha \gamma^l \psi_\beta = 0$$

(since $\gamma^\mu\psi_\mu = 0$), we derive

$$B = \Re\left(i\overline{\delta\psi}_\rho \gamma^c \psi_\mu (\det e) e_p^\rho e_q^\mu e_r^\nu\right)\epsilon^{dpqr}\epsilon_{dabc}\widehat{\omega}_\nu^{ab} =$$

$$= \Re\left(i\left(\overline{\delta\psi}_\rho \gamma^\nu \psi_\mu - \overline{\delta\psi}_\mu \gamma^\nu \psi_\rho\right)\right) e_a^\rho e_b^\mu \widehat{\omega}_\nu^{ab} \det e =$$

$$= i\delta(\overline{\psi}_\rho \gamma^\nu \psi_\mu) e_a^\rho e_b^\mu \widehat{\omega}_\nu^{ab} \det \mathrm{e} = i\delta(\overline{\psi}_\rho\gamma^\nu\psi_\mu) g^{\rho[\lambda} g^{\kappa]\mu} g_{\nu\varepsilon} E_{\lambda\varepsilon\kappa} \det e =$$

$$= \frac{1}{4} i\delta(\overline{\psi}_\rho\gamma^\nu\psi_\mu)\epsilon^{\rho\mu\alpha\beta}\epsilon_{\sigma\delta\alpha\beta} g^{\lambda\sigma} g^{\kappa\delta} g_{\nu\varepsilon} E_{\lambda\varepsilon\kappa} \det e =$$

$$= -\frac{1}{4\det e} i\delta(\overline{\psi}_\rho\gamma^\nu\psi_\mu)\epsilon^{\rho\mu\alpha\beta}\epsilon^{\lambda\kappa\sigma\delta} g_{\alpha\sigma} g_{\beta\delta} g_{\nu\varepsilon} E_{\lambda\varepsilon\kappa} = \frac{1}{4}(P+N+M)$$

Using again later $\gamma^\mu\psi_\mu = 0$ and (6) we will have

$$2M=\frac{i}{\det e}\delta(\overline{\psi}_\rho\gamma^\nu\psi_\mu)g_{\alpha\sigma}g_{\beta\delta}\epsilon^{\rho\mu\alpha\beta}\epsilon^{\lambda\kappa\sigma\delta}g_{\nu\varepsilon}W_{\kappa\varepsilon\lambda}=$$
$$=\frac{k_N^2}{8(\det e)^2}\delta(\overline{\psi}_\rho\gamma_\varepsilon\psi_\mu)g_{\alpha\sigma}g_{\beta\delta}\epsilon^{\rho\mu\alpha\beta}\epsilon^{\lambda\kappa\sigma\delta}\epsilon_{dabc}(\overline{\psi}_\nu\gamma^c\psi_\eta)e_\gamma^d e_\kappa^a e_\lambda^b\epsilon^{\eta\nu\varepsilon\gamma}=$$
$$=\frac{k_N^2}{2\det e}\delta(\overline{\psi}_\rho\gamma_\varepsilon\psi_\mu)g_{\alpha\sigma}g_{\beta\delta}\epsilon^{\rho\mu\alpha\beta}(\overline{\psi}_\nu\gamma^c\psi_\eta)\epsilon^{\eta\nu\varepsilon[\delta}e_c^{\sigma]}=$$
$$=-\frac{k_N^2}{2}(\det e)\delta(\overline{\psi}^\lambda\gamma_\varepsilon\psi^\kappa)(\overline{\psi}_\nu\gamma^\sigma\psi_\eta)\epsilon_{\sigma\delta\lambda\kappa}\epsilon^{\eta\nu\varepsilon\delta}=$$
$$=-k_N^2(\det e)\delta(\overline{\psi}^{j\lambda}\gamma^\varepsilon\psi^{j\kappa})(\overline{\psi}_\kappa^l\gamma_\varepsilon\psi_\lambda^l)=$$
$$=\frac{k_N^2}{2}(\det e)\delta((\psi^{j\lambda}C\gamma^\varepsilon\psi^{j\kappa})(\psi_\lambda^l C\gamma_\varepsilon\psi_\kappa^l))$$
$$P=-\frac{i}{\det e}\delta(\overline{\psi}_\rho\gamma^\nu\psi_\mu)g_{\kappa\varepsilon}g_{\sigma\alpha}g_{\delta\beta}\epsilon^{\rho\mu\alpha\beta}\epsilon^{\lambda\kappa\sigma\delta}W_{\lambda\varepsilon\nu}=$$
$$=\frac{k_N^2}{8}(\det e)\delta(\overline{\psi}_\rho\gamma^\nu\psi_\mu)\epsilon^{\rho\mu\alpha\beta}\epsilon_{\kappa\varepsilon\alpha\beta}g^{\kappa\lambda}\epsilon_{dabc}(\overline{\psi}_\eta\gamma^c\psi_\delta)e_\gamma^d e_\lambda^a e_\nu^b\epsilon^{\delta\eta\varepsilon\gamma}=$$
$$=\frac{k_N^2}{2}(\det e)\delta(\overline{\psi}_{[\rho}\gamma^b\psi_{\mu]})e^{\rho a}\epsilon_{dabc}\epsilon^{\delta\eta\mu\gamma}e_\gamma^d(\overline{\psi}_\eta\gamma^c\psi_\delta)=$$
$$=-\frac{k_N^2}{2}(\det e)\delta(\overline{\psi}_{[\rho}\gamma^b\psi_{\mu]})e^{\rho a}e_p^\delta e_q^\eta e_r^\mu\epsilon^{dpqr}\epsilon_{dabc}(\overline{\psi}_\eta\gamma^c\psi_\delta)=$$
$$=\frac{k_N^2}{2}(\det e)\delta((\psi^{j\delta}C\gamma^\eta\psi^{j\mu})(\psi_\delta^l C\gamma_\mu\psi_\eta^l))$$
$$N=\frac{i}{\det e}\delta(\overline{\psi}_\rho\gamma^\nu\psi_\mu)\epsilon^{\rho\mu\alpha\beta}\epsilon^{\lambda\kappa\sigma\delta}g_{\alpha\sigma}g_{\beta\delta}g_{\nu\kappa}W_{\lambda\varepsilon\varepsilon}=$$
$$=-(\det e)i\delta(\overline{\psi}_\rho\gamma^\nu\psi_\mu)\epsilon^{\rho\mu\alpha\beta}\epsilon_{\kappa\nu\alpha\beta}g^{\kappa\lambda}W_{\lambda\varepsilon\varepsilon}=$$
$$=-2(\det e)i(\delta(\overline{\psi}^\lambda\gamma^\mu\psi_\mu)-\delta(\overline{\psi}_\rho\gamma^\rho\psi^\lambda))W_{\lambda\varepsilon\varepsilon}=0 \quad (\text{since } \gamma^\mu\psi_\mu=0\ )$$

Therefore $B=-\frac{1}{4}(\det e)\delta(2B_1(\psi^\delta,\psi_\delta)+B_2(\psi^\delta,\psi_\delta))$,

where for $\epsilon=(\epsilon^1,\epsilon^2)$, $\epsilon'=(\epsilon'^1,\epsilon'^2)$ with ϵ^j,ϵ'^j anticommuting Majorana spinors (they anticommute also with ψ_μ^l) we define :

$$B_1(\epsilon,\epsilon')=i\frac{k_N}{2}(\overline{\epsilon}^j\gamma^\mu\psi^{j\eta})i\frac{k_N}{2}(\overline{\epsilon}'^l\gamma_\eta\psi_{j\mu})\ ,$$
$$B_2(\epsilon,\epsilon')=i\frac{k_N}{2}(\overline{\epsilon}^j\gamma^\mu\psi^{j\eta})i\frac{k_N}{2}(\overline{\epsilon}'^l\gamma_\mu\psi_\eta^l)\ .$$

For a square matrix $(h_{ij})_{i,j}$ we have $\det h = \exp(\operatorname{tr}\log h)$ and so $\delta \det h = (\det h)\operatorname{tr}(h^{-1}\delta h)$. In particular we have $\delta \det g = (\det g) g^{\mu\lambda}\delta g_{\mu\lambda}$, $\delta \det e = (\det e) e_d^{\nu}\delta e_{\nu}^{d}$.

For a given ϵ we take $e_{\nu}^{d}(\varepsilon) = e_{\nu}^{d}(0) + i\frac{k_N}{2}(\bar{\epsilon}^j \varepsilon \gamma^d \psi_{\nu}^{j})$ and then ,

with $\varepsilon \in \mathbb{R}$ we have $B_1(\epsilon,\epsilon) = e_a^{\mu}\frac{d e_{\eta}^{a}}{d\varepsilon} e_b^{\eta}\frac{d e_{\mu}^{b}}{d\varepsilon}\bigg|_{\varepsilon=0} = -\frac{d e_b^{\mu}}{d\varepsilon}\frac{d e_{\mu}^{b}}{d\varepsilon}\bigg|_{\varepsilon=0} =$

$$= \frac{1}{2}\left(e_b^{\mu}\frac{d^2 e_{\mu}^{b}}{d\varepsilon^2} + e_{\mu}^{b}\frac{d^2 e_b^{\mu}}{d\varepsilon^2} - \frac{d^2}{d\varepsilon^2}(e_b^{\mu} e_{\mu}^{b}) \right)\bigg|_{\varepsilon=0} = \frac{1}{2} e_{\mu}^{b}\frac{d^2 e_b^{\mu}}{d\varepsilon^2}\bigg|_{\varepsilon=0} \qquad (17)$$

For any $\varepsilon \in \mathbb{R}$ we have $e_{\mu}^{b}\frac{d e_b^{\mu}}{d\varepsilon} = -e_b^{\mu}\frac{d e_{\mu}^{b}}{d\varepsilon} = 0$ (because $\gamma^{\mu}\psi_{\mu} = 0$)

and therefore $e_{\mu}^{b}\frac{d^2 e_b^{\mu}}{d\varepsilon^2} = -\frac{d e_{\mu}^{b}}{d\varepsilon}\frac{d e_b^{\mu}}{d\varepsilon}$ and considering (17) it follows

$B_1(\epsilon,\epsilon) = 0$. (18)

In a similar way we have $B_2(\epsilon,\epsilon) = \frac{d e_{\lambda}^{a}}{d\varepsilon}\frac{d e_{\mu}^{b}}{d\varepsilon} g^{\lambda\mu}\eta_{ab}\bigg|_{\varepsilon=0} =$

$$= \frac{1}{2}\frac{d^2 g_{\lambda\mu}}{d\varepsilon^2} g^{\lambda\mu}\bigg|_{\varepsilon=0} = g_{\lambda\mu}\frac{d e_c^{\lambda}}{d\varepsilon}\frac{d e_d^{\mu}}{d\varepsilon}\eta^{cd}\bigg|_{\varepsilon=0} = \frac{1}{2} g_{\lambda\mu}\frac{d^2 g^{\lambda\mu}}{d\varepsilon^2}\bigg|_{\varepsilon=0} \qquad (19)$$

(where for the last equality we used (17) and (18))

We have now at $\varepsilon = 0$ again using (17), (18), $\frac{d^2 e_{\lambda}^{a}}{d\varepsilon^2} = 0$,(19) and

$e_{\lambda}^{a}\frac{d e_c^{\lambda}}{d\varepsilon} = -e_c^{\lambda}\frac{d e_{\lambda}^{a}}{d\varepsilon}$ that $0 = \frac{d^2}{d\varepsilon^2}(e_{\lambda}^{a} e_{\mu}^{b} e_c^{\lambda} e_d^{\mu}\eta_{ab}\eta^{cd}) = 2B_2(\epsilon,\epsilon)$ and so

$B_2(\epsilon,\epsilon) = 0$ (20).

Because the $\epsilon, \epsilon', \psi$ components all anticommute we have

$B_j(\epsilon,\epsilon') = B_j(\epsilon',\epsilon)$ and so

$$B_j(\epsilon,\epsilon') = \frac{1}{2}(B_j(\epsilon+\epsilon',\epsilon+\epsilon') - B_j(\epsilon,\epsilon) - B_j(\epsilon',\epsilon')) = 0$$

Hence *A*= 0 and the additional variation of the action under $\delta\psi_{\mu} = i g\gamma_{\mu}\epsilon$ variation of the gravitinos is linear in ψ.

Thus obviously $\delta S = \int F(e, \partial e, \psi, \partial\psi, \epsilon, \partial\epsilon) d^4 x$ with F real and linear in ψ and ϵ considered as functions.

Notice that although it is a dependence on the gauge field in *F*, it carries a factor of g^2 and so it can be neglected.
Performing a discretization of the fields, converting all derivatives of a field from the expression of *F* into linear matrix operators acting on the respective field , having space-time dependent coefficients that are determined by discretization parameters we reduce *F* to a form $\bar{F}=\Re F_{\mu\alpha\beta}(x,e)\bar{\psi}_{\mu\alpha}\epsilon^{\beta}$ with μ a covariant index and α,β spinor indexes. Since $\bar{F}$ must be a scalar invariant under Lorentz transformations, invariant under general orientation preserving coordinate transforms and changes sign under orientation changing coordinates transforms we conclude that it must have an expression as $\bar{F}=(\det e)\Re K(x)e_d^{\nu}\bar{\psi}_{\mu\alpha}P^{\alpha}_{\beta}\gamma^{d}_{\beta\delta}\epsilon^{\delta}$, Lorentz invariance requiring that $S^{-1}PS=P$ for any $S=S(\bar{\Lambda})$ and $K(\bar{\Lambda}x)=K(x)$ with $\bar{\Lambda}=\exp(\lambda^{ab}J_{ab})$, $S=S(\bar{\Lambda})=\exp(\frac{1}{2}\lambda^{ab}\sigma_{ab})$ which leads to

$$\delta S=\int d^4x(\det e)e_d^{\nu}\Re(\bar{\Lambda}_1\bar{\psi}_{\nu}\gamma_5\gamma^d\epsilon+\bar{\Lambda}_2 i\frac{k_N}{2}\bar{\psi}_{\nu}\gamma^d\epsilon).$$

The Λ_1 and Λ_2 are constants that come from the conversion of differential operators into linear matrix operators determined by discretization parameters. The Λ_1 term changes sign under a parity transformation of the Lorentz indices combined with a parity in the covariant indices and so it must be disgarded. Also the action variation contains only terms of the form $\psi^1_{\mu}...\epsilon^1$ or $\psi^2_{\nu}...\epsilon^2$ and so $\bar{\Lambda}_2=-\Lambda\in\mathbb{R}$.
Λ cosmological constant as we will see.

Since $\gamma^{\nu}\psi_{\nu}=0$ restriction is imposed the variation δS vanishes automatically as it follows from the derived (21) form.
We notice that if we not impose the restriction $\gamma^{\nu}\psi_{\nu}=0$, the other δS terms containing $\gamma^{\nu}\psi_{\nu}$ (for the full gauged supergravity variations $\delta\psi_{\mu}=\frac{1}{k_N}(\tilde{D}_{\mu}\epsilon+ig\gamma_{\mu}\epsilon)$, $\delta e^d_{\nu}=\frac{k_N}{2}\Re(i\bar{\epsilon}\gamma^d\psi_{\nu})$) are cubic in ψ and so if we still want invariance under the above variations for small fermionic perturbations we must add to the gauged supergravity action a cosmological constant term $S_{\Lambda}=\int\Lambda(\det e)d^4x$ which obviously vanishes if $\gamma^{\nu}\psi_{\nu}=0$ since $\delta(\det e)=(\det e)e_d^{\nu}\delta e^d_{\nu}=0$.

The dependence of Λ on the discretization parameters is consistent with the estimation of the vacuum energy density and therefore of the cosmological constant using the Planck units as discretization parameters. (see Vacuum energy density estimation article).

If we would consider variations

$\delta\psi_\mu=\frac{1}{k_N}(g\gamma_\mu\epsilon+D_\mu\epsilon)$, $\delta e_\nu^d=i\frac{k_N}{2}\bar{\epsilon}\gamma^d\psi_\nu$ in the case of only one ungauged gravitino field, since $i\gamma_\mu\epsilon$ is no more a Majorana spinor, if ϵ is, we must take $g\in\mathbb{R}$.

The restriction $\gamma^\mu\psi_\mu=0$ imposes also restrictions on ϵ because we must have $\delta(\gamma^\mu\psi_\mu)=(\delta\gamma^\mu)\psi_\mu+\gamma^\mu\delta\psi_\mu=0$ and so

$$i\frac{k_N}{2}(\bar{\epsilon}\gamma^d\psi_\mu)\gamma_d\psi^\mu+\frac{1}{k_N}\gamma^\mu D_\mu\epsilon+\frac{4}{k_N}g\epsilon=0 \quad (21)$$

ϵ , ψ_μ being anticommuting Majorana spinors.

As we proved above , the application $B:E\times E\ni(\epsilon_1,\epsilon_2)\rightarrow(\overline{\epsilon_1}\gamma^d\psi_\mu)(\overline{\epsilon_2}\gamma_d\psi^\mu)\in\mathbb{R}$ is symmetric bilinear and satisfies $B(\epsilon,\epsilon)=0$ on the space E of anticommuting (with ψ_μ also) Majorana spinors and so $B\equiv0$ and therefore $(\bar{\epsilon}\gamma^d\psi_\mu)\gamma_d\psi^\mu=0$ for $\epsilon\in E$. Hence if $\epsilon\in E$ satisfies (21) then

$$\gamma^\mu D_\mu\epsilon+4g\epsilon=0 \quad (22).$$

Under a general coordinate transform $x^\mu\rightarrow x'^\mu=x^\mu+\xi^\mu(x)$ the vielbein and the spin connection coefficients transform as

$$e_a^\lambda(x)\rightarrow e'^\lambda_a(x')=\rightarrow\frac{\partial\xi^\lambda}{\partial x^\mu}(x)e_a^\mu(x)+e_a^\lambda(x)$$

$$\omega_\mu^{ab}(x)\rightarrow\omega'^{ab}_\mu(x')=\omega_\mu^{ab}(x)-\frac{\partial\xi^\lambda}{\partial x'^\mu}(x')\omega_\lambda^{ab}(x)$$

$$\epsilon(x)\rightarrow\epsilon'(x')=\epsilon(x)$$

and under a local Lorentz transformation $\Lambda=(\Lambda_b^a(x))_{a,b}=\exp(\lambda^{ab}J_{ab})$ $S(\Lambda)=\exp(\frac{1}{2}\lambda^{ab}\sigma_{ab})$, $\lambda^{ab}=-\lambda^{ba}=\lambda^{ab}(x)$ transform as

$$e_a^\lambda(x)\rightarrow e'^\lambda_a(x)=\eta_{ac}\eta^{bd}\Lambda_d^c(x)e_b^\lambda(x)$$

$$\omega_\mu^{ab}(x)\rightarrow\omega'^{ab}_\mu(x)=\partial_\mu\Lambda_c^a(x)\eta^{cd}\Lambda_d^b(x)+\Lambda_c^a(x)\omega_\mu^{cd}\Lambda_d^b(x)$$

$$\epsilon(x)\rightarrow S(\Lambda)\epsilon(x)$$

We can obviously at any point $x_0 \in \mathbb{R}^4$ choose arbitrary values for $\partial_\lambda \xi^\mu(x_0)$, $\partial_\nu \partial_\lambda \xi^\mu(x_0)$, $\lambda^{ab}(x_0)$, $\partial_\mu \lambda^{ab}(x_0)$, $\partial_\mu \partial_\nu \lambda^{ab}(x_0)$ and so for any $x_0 \in \mathbb{R}^4$ we can take $\xi = \xi(x)$, $\Lambda = \Lambda(x)$ such that $e'^a_\lambda = \delta^a_\lambda + O(\varepsilon)$, $\partial_\nu e'^a_\lambda = O(\varepsilon)$, $\omega'^{ab}_\mu = O(\varepsilon)$, $\partial_\nu \omega'^{ab}_\mu = O(\varepsilon)$ on a small neighborhood D_ε of x_0 with $\operatorname{diam} D_\varepsilon = \varepsilon$.

If $\epsilon = \epsilon(x)$ satisfies (22) for ω and e as spin connection respective vielbein, then $\epsilon' = \epsilon'(x')$ satisfies (22) with ω', e' as spin connection and vielbein and redefining ϵ', ω', e' as $\epsilon, \omega, e,$ $\epsilon = \epsilon(x)$ we have a Majorana spinor field ϵ that staisfies (22) and for $x_0 \in \mathbb{R}^4$, on the small neighborhood D_ε of x_0 we have

$\gamma^\mu \partial_\mu \epsilon + 4 g \epsilon = A_\varepsilon(x)$ (23) with $\partial_\lambda A_\varepsilon(x) = O(\varepsilon)$, $A_\varepsilon(x) = O(\varepsilon)$ for $x \in D_\varepsilon$ and in (23) , $(\gamma^\mu)_\mu$ are identical the Dirac gamma matrices $(\gamma^a)_a$

Let $\varphi = \varphi(x) = \begin{cases} \epsilon(x) & \text{for } x \in D_\varepsilon \\ 0 & \text{for } x \in \mathbb{R}^4 \setminus D_\varepsilon \end{cases}$, $V_\varepsilon = \int\limits_{D_\varepsilon} d^4 x$,

$$u_\varepsilon(p) = \frac{1}{(2\pi)^4} \int \exp(-ip\bar{x}) \varphi(x) d^4 x = \frac{1}{(2\pi)^4} \int\limits_{D_\varepsilon} \exp(-i p \bar{x}) \epsilon(x) d^4 x$$

(24)

with $\bar{x} = x - x_0$. Then we have

$\varphi(x) = \int \exp(i p \bar{x}) u_\varepsilon(p) d^4 p =$

$= \int\limits_{p_0 > 0} (\exp(i p \bar{x}) u_\varepsilon(p) + \exp(-i p \bar{x}) u_\varepsilon(p)) d^4 p$, for $x \in D_\varepsilon$, $p = (p_\mu)_\mu$

$u_\varepsilon(p)$ is a spinor and decomposes as $u_\varepsilon = u_1 + u_2$ with $u_1, i u_2$ Majorana spinors:

$u_1^T(p) = (a(p), b(p), b^*(p), -a^*(p))$

$u_2^T(p) = (i c(p), i d(p), i d^*(p), -i c^*(p))$

$u_1^T(-p) = (a'(p), b'(p), b'^*(p), -a'^*(p))$

$u_2^T(-p) = (i c'(p), i d'(p), i d'^*(p), -i c'^*(p))$

for $p_0 > 0$.

The fact that $\varphi(x)$ is a Majorana spinor translates into

$i\gamma^0 C^T \varphi(x) = \varphi^*(x)$, $i\gamma^0 C^T u_\varepsilon(p) = u_\varepsilon^*(-p)$, $i\gamma^0 C^T u_\varepsilon(-p) = u_\varepsilon^*(p)$

for $p_0 > 0$ which leads to $a = a', b = b', c = -c', d = -d'$

$$\varphi(x)=\int_{p_0>0} d^4 p\left(\left(\begin{matrix}\exp(\ \ i p\bar{x})\begin{pmatrix}a & +ic\\ b & +id\end{pmatrix}\\ \exp(-ip\bar{x})\begin{pmatrix}b^*-id^*\\ -a^*+ic^*\end{pmatrix}\end{matrix}\right)+\left(\begin{matrix}\exp(-ip\bar{x})\begin{pmatrix}a & -ic\\ b & -id\end{pmatrix}\\ \exp(\ \ ip\bar{x})\begin{pmatrix}b^*+id^*\\ a^*-ic^*\end{pmatrix}\end{matrix}\right)\right)$$

For $\bar{p}=(\bar{p}_0,0,0,0)$, $\bar{p}_0>0$ we obtain

$$\frac{1}{(2\pi)^4}\int\exp(-i\bar{p}\bar{x})(\gamma^\mu\partial_\mu\varphi+4g\varphi)d^4x=$$

$$=i\bar{p}_0\gamma^0 u_\varepsilon(\bar{p})+4g u_\varepsilon(\bar{p})=\int_{D_\varepsilon}\frac{1}{(2\pi)^4}\exp(-i\bar{p}\bar{x})A_\varepsilon(x)d^4x$$

$$\begin{pmatrix}\bar{p}_0(b^*+id^*)+4g(a+ic)\\ \bar{p}_0(a^*-ic^*)+4g(b+id)\\ \bar{p}_0(a+ic)+4g(b^*+id^*)\\ \bar{p}_0(b+id)+4g(a^*-ic^*)\end{pmatrix}=\frac{1}{(2\pi)^4}\int\exp(-i\bar{p}\bar{x})A_\varepsilon(x)d^4x$$

and for $\bar{p}=(-\bar{p}_0,0,0,0)$, $\bar{p}_0>0$ we obtain

$$\begin{pmatrix}-\bar{p}_0(b^*-id^*)+4g(a-ic)\\ -\bar{p}_0(-a^*+ic^*)+4g(b-id)\\ -\bar{p}_0(a-ic)+4g(b^*-id^*)\\ -\bar{p}_0(b-id)+4g(-a^*+ic^*)\end{pmatrix}=\frac{1}{(2\pi)^4}\int_{D_\varepsilon}\exp(-i\bar{p}\bar{x})A_\varepsilon(x)d^4x$$

leading to

$$8gic=2\bar{p}_0b^*+O(\varepsilon)V_\varepsilon \quad , \quad 8gid=O(\varepsilon)V_\varepsilon \ ,$$
$$8gid=2\bar{p}_0a+O(\varepsilon)V_\varepsilon \quad , \quad 2\bar{p}_0id=O(\varepsilon)V_\varepsilon$$

and so we proved that

$$u_\varepsilon^T(\bar{p})=(ic(\bar{p}),-ic^*(\bar{p})4\frac{g}{\bar{p}_0},ic(\bar{p})4\frac{g}{\bar{p}_0},-ic^*(\bar{p}))+O(\varepsilon)V_\varepsilon$$

for $\bar{p}=(\bar{p}_0,0,0,0)$, $\bar{p}_0>0$

and

$$u_\varepsilon^T(\bar{p})=(-ic(-\bar{p}),-ic^*(-\bar{p})4\frac{g}{\bar{p}_0},ic(-\bar{p})4\frac{g}{\bar{p}_0},ic^*(-\bar{p}))+O(\varepsilon)V_\varepsilon$$

for $\bar{p}=(-\bar{p}_0,0,0,0)$, $\bar{p}_0>0$

From the (24) relation follows $\lim_{\varepsilon\to 0} \frac{u_\varepsilon(\bar{p})}{V_\varepsilon} = \frac{1}{(2\pi)^4}\epsilon(x_0)$ and since x_0 is arbitrary, we conclude that ϵ has the form $\epsilon^T = (0, B, B^*, 0)$ with $B = B(x)$.

The (23) system now becomes

$$\partial_0 B^* + \partial_3 B^* = i A_\varepsilon^0$$

$$\partial_1 B^* + i\partial_2 B^* + 4 g i B = i A_\varepsilon^1 \qquad (25)$$

$$\partial_1 B - i\partial_2 B - 4 i g B^* = -i A_\varepsilon^2 \qquad (26)$$

$$\partial_0 B + \partial_3 B = i A_\varepsilon^3$$

From (25) and (26) follows now

$$\frac{1}{(2\pi)^4}\int d^4 x \exp(-i p x)(\partial_1 B - \partial_2 B - 4 i g B^*) =$$

$$= (i p_1 + p_2) u_\varepsilon^1(p) - 4 i g u_\varepsilon^3(p) = O(\varepsilon) V_\varepsilon$$

$$(i p_1 - p_2) u_\varepsilon^3(p) + 4 i g u_\varepsilon^1(p) = O(\varepsilon) V_\varepsilon$$

$(p_1^2 + p_2^2 + 16 g^2) u_\varepsilon^1(p) = O(\varepsilon) V_\varepsilon$ which leads to

$$\frac{1}{(2\pi)^4} B(x_0) = \lim_{\varepsilon\to 0} \frac{u_\varepsilon^1(p)}{V_\varepsilon} = 0 \ .$$

Since x_0 is arbitrary we conclude $\epsilon \equiv 0$ or $g = 0$.

Hence the (22) equation is not a suitable equation with *g* real non-zero for Majorana spinors and a cosmological constant can arise naturally only in gauged supergravity , imposing invariance of the action under the additional supersymmetry variation

$$\delta\psi_\mu = i g \gamma_\mu \epsilon \ , \ \psi_\mu = \psi_\mu^1 + i\psi_\mu^2 \ , \ \epsilon = \epsilon^1 + i\epsilon^2 \ .$$

So finally we have invariance of the gauged supergravity action with cosmological constant term under the field variations

$$\delta e_v^d = \frac{k_N}{2} i \epsilon^j \gamma^d \psi_v^j \ ,$$

$$\delta\psi_\mu = \frac{1}{k_N}(D_\mu \epsilon + i g A_\mu \epsilon + i g \gamma_\mu \epsilon) \ .$$

References

[1] Horațiu Năstase – Introduction to the AdS/CFT correspondence,
Cambridge University Press
[2] A.Zee – Quantum Field Theory in a Nutshell,
Princeton University Press

[3] Laurent Freidel, Jerzy Kowalski-Glikman, Robert G. Leigh,
Djordje Minic – The Vacuum energy Density and Gravitational
Entropy